Example : 1

(i) Find the 7th term in the expansion of $\left(\frac{4x}{5}-\frac{5}{2x}\right)^{9}$

(ii) Find the coefficient of x^7 in $\left(ax^{2}+\frac{1}{bx}\right)^{11}$

Solution

(i) In the expansion of $\left(\frac{4x}{5}-\frac{5}{2x}\right)^{9}$

The general terms is $T_{r+1} = {}^{9}C_{r}\left(\frac{4x}{5}\right)^{9-r}\left(-\frac{5}{2x}\right)^{r}$

For 7th term (T_7), Put r = 6

$$\Rightarrow \quad T_7 = T_{6+1} = {}^{9}C_{6}\left(\frac{4x}{5}\right)^{9-6}\left(-\frac{5}{2x}\right)^{6}$$

$$\Rightarrow \quad T_7 = \frac{9\times 8\times 7}{3!}\left(\frac{4}{5}\right)^{3} x^{3}\left(-\frac{5}{2}\right)^{6}\frac{1}{x^{6}}$$

$$\Rightarrow \quad T_7 = \frac{9\times 8\times 7}{3!}\, 5^{3}\,\frac{1}{x^{3}}$$

$$\Rightarrow \quad T_7 = \frac{10500}{x^{3}}$$

(ii) In $\left(ax^{2}+\frac{1}{bx}\right)^{11}$ general term is $T_{r+1} = {}^{11}C_{r}\, a^{11-r}\, b^{-r}\, x^{22-3r}$

for term involving x^7 , $22 - 3r = 7$

$\Rightarrow \quad r = 5$

Hence T_{5+1} or the 6th term will contain x^7.

$$T_6 = {}^{11}C_{5}\,(ax^{2})^{11-5}\left(\frac{1}{bx}\right)^{5} = \frac{11\times 10\times 9\times 8\times 7}{5!}\,\frac{a^{6}}{b^{5}}\,x^{7} = \frac{462a^{6}}{b^{5}}\,x^{7}$$

Hence the coefficient of x^7 is $\frac{462a^{6}}{b^{5}}$

Example : 2

Find the term independent of x in $\left(\frac{3x^{2}}{2}-\frac{1}{3x}\right)^{9}$

Solution

$$T_{r+1} = {}^{9}C_{r}\left(\frac{3x^{2}}{2}\right)^{9-r}\left(-\frac{1}{3x}\right)^{r} = {}^{9}C_{r}\left(\frac{3x^{2}}{2}\right)^{9-r}\left(-\frac{1}{3x}\right)^{r} x^{18-3r}$$

for term independent of x, $18 - 3r = 0$

$\Rightarrow \quad r = 6$

Hence T_{6+1} or 7th term is independent of x.

$$T_7 = {}^{9}C_{6}\left(\frac{3x^{2}}{2}\right)^{9-6}\left(-\frac{1}{3x}\right)^{6} = \frac{9\times 8\times 7}{3!}\left(\frac{3}{2}\right)^{3}\left(-\frac{1}{3}\right)^{6} = \frac{7}{18}$$

Example : 3

Find the coefficient of x^{11} in the expansion of $(2x^2 + x - 3)^6$.

Solution

$(2x^2 + x - 3)^6 = (x - 1)^6 (2x + 3)^6$

term containing x^{11} in $(2x^2 + x - 3)^6$

$(x - 1)^6 = {}^6C_0\, x^6 - {}^6C_1\, x^5 + {}^6C_2\, x^4 - {}^6C_3\, x^3 + \ldots\ldots\ldots\ldots$

$(2x + 3)^6 = {}^6C_0\, (2x)^6 + {}^6C_1\, (2x)^5\, 3 + {}^6C_2\, (2x)^4\, 3^2 + \ldots\ldots\ldots$

term containing x^{11} in the product $(x - 1)^6 (2x + 3)^6 = [C_0\, x^6]\, [{}^6C_1\, (2x)^5\, 3] - [{}^6C_1\, x^5]\, [{}^6C_0\, (2x)^6\,]$

$= 32\, (18\, x^{11}) - 6\, (64)\, x^{11} = 192\, x^{11}$

$\Rightarrow$ the coefficient of x^{11} is 192

Example : 4

Find the relation between r and n so that coefficient of 3r^th^ and (r + 2)^th^ terms of $(1 + x)^{2n}$ are equal.

Solution

In $(1 + x)^n$, $T_{r+1} = {}^{2n}C_r\, x^r$

$T_{3r} = {}^{2n}C_{3r-1}\, x^{3r-1}$

$T_{r+2} = {}^{2n}C_{r+1}\, x^{r+1}$

If the coefficient are equal then ${}^{2n}C_{3r-1} = {}^{2n}C_{r+1}$

There are two possibilities

Case – 1

$3r - 1 = r + 1$

$\Rightarrow$ $r = 1$

$\Rightarrow$ $T_{3r} = T_3$ and $T_{r+2} = T_3$

$\Rightarrow$ T_{3r} and T_{r+2} are same terms

Case – 2

${}^{2n}C_{3r-1} = {}^{2n}C_{r+1}$

$\Rightarrow$ ${}^{2n}C_{3r-1} = {}^{2n}C_{2n-(r+1)}$

$\Rightarrow$ $3r - 1 = 2n - (r + 1)$

$\Rightarrow$ $r = n/2$

Example : 5

Find the coefficient of x^3 in the expansion $(1 + x + x^2)^n$.

Solution

$(1 + x + x^2)^n = [1 + x\, (1 + x)]^n = {}^nC_0 + {}^nC_1\, x\, (1 + x) + {}^nC_2\, x^2\, (1 + x)^2 + \ldots\ldots\ldots\ldots$

Coefficient of $x^3 = {}^nC_2$ [coeff of x in $(1 + x)^2$] $+ {}^nC_3$ [coeff of x^0 in $(1 + x)^3$]

$$= {}^nC_2\, (2) + {}^nC_3\, (1) = \frac{2n(n-1)}{2} + \frac{n(n-1)(n-2)}{3!} = \frac{n(n-1)}{6}\,[6 + n - 2] = \frac{n(n-1)(n+4)}{6}$$

Example : 6

If ${}^nC_r$ is denoted as C_r, show that

(a) $$(C_0 + C_1)\,(C_1 + C_2)\,(C_2 + C_3)\ \ldots\ldots\ldots\ (C_{n-1} + C_n) = \frac{C_0 C_1 \ldots . C_n (n+1)^n}{n!}$$

(b) $$\frac{C_1}{C_0} + 2\,\frac{C_2}{C_1} + 3\,\frac{C_3}{C_2} + \ldots\ldots . + n\,\frac{C_n}{C_{n-1}} = \frac{n(n+1)}{2}$$

Solution

(a) LHS $= (C_0 + C_1)\,(C_1 + C_2)\,(C_2 + C_3)\ \ldots\ldots\ (C_{n-1} + C_n)$

Multiply and Divide by $C_0\, C_1\, C_2\, \ldots . C_n = C_0\, C_1\, C_2\, \ldots\ldots C_n \left(1+\frac{C_1}{C_0}\right)\left(1+\frac{C_2}{C_1}\right) \ldots\ldots \left(1+\frac{C_n}{C_{n-1}}\right)$

$$\text{using } \frac{C_r}{C_{r-1}} = \frac{n-r+1}{r} = C_0\, C_1\, C_2\, C_3 \ldots\ldots C_n \left(1+\frac{n-1+1}{1}\right) \times \left(1+\frac{n-2+1}{2}\right) + \ldots\ldots\ldots + \left(1+\frac{n-n+1}{n}\right)$$

$$= C_0\, C_1\, C_2 \ldots\ldots C_n \left(\frac{n+1}{1}\right)\left(\frac{n+1}{2}\right) + \ldots\ldots . + \left(\frac{n+1}{n}\right) = C_0 C_1 C_2 C_3 \ldots\ldots C_n \frac{(n+1)^n}{n!} = \text{RHS}$$

(b) LHS = $\frac{C_1}{C_0} + 2\,\frac{C_2}{C_1} + 3\,\frac{C_3}{C_2} + \text{.........} + n\,\frac{C_n}{C_{n-1}}$

using $\frac{C_r}{C_{r-1}} = \frac{n-r+1}{r} = \left(\frac{n-1+1}{1}\right) + 2\left(\frac{n-2+1}{2}\right) + \text{.......} + n\,\frac{(n-n+1)}{n}$

= n + (n – 1) + (n – 2) + ... + 1

= Sum of first n natural numbers = $\frac{n(n+1)}{2}$ = RHS

Example : 7

Show that

(a) $C_0^2 + C_1^2 + C_2^2 + C_3^2 + \text{.........} + C_n^2 = \frac{(2n)!}{n!\,n!}$

(b) $C_0\,C_1 + C_1C_2 + C_2C_3 + \text{......}C_{n-1}\,C_n = \frac{(2n)!}{(n-1)!\,(n+1)!}$

Solution

Consider the identities $(1 + x)^n = C_0 + C_1 x + C_2x^2 + \text{...........} + C_nx^n\ (1 + x)^n$

$= C_0x^n + C_1 X^{n-1} + C_2\,x^{n-2} + \text{.......} + C_n$

multiplying these we get another identity

$(1 + x)^n\,(x + 1)^n = (C_0 + C_1x = (C_0 + C_1x + C_2x^2 + \text{........} + C_nx^n) = C_0x^n + C_1x^{n-1} + C_2x^{n-2} + \text{.......} + C_n)$

(a) Compare coefficients of x^n on both sides

In LHS, coeff. of x^n = coeff of x^n in $(1 + x)^{2n} = {}^{2n}C_0$

In RHS, terms containing x^n are $C_0^{\,2}\,x^n + C_1^{\,2}\,x^n + C_2^{\,2}\,x^n + \text{.......} + C_n^{\,2}\,x^n$

$\Rightarrow$ Coeff. of x^n on RHS = $C_0^{\,2} + C_1^{\,2} + C_2^{\,2} + \text{.......} + C_n^{\,2}$

equating the coefficients $C_0^{\,2} + C_1^{\,2} + C_2^{\,2} + \text{.........}\ C_n^{\,2} = {}^{2n}C_n$

$$C_0^2 + C_1^2 + C_2^2 + \text{.........} + C_n^2 = \frac{(2n)!}{n!\,n!}$$

(b) Compare the coefficients of x^{n-1} on both sides

In LHS, coeff. of $x^{n-1} = {}^{2n}C_{n-1}$

In RHS, term containing x^{n-1} is $C_0C_1\,x^{n-1} + C_1C_2\,x^{n-1} + \text{........}$

Hence coeff. of x^{n-1} in RHS = $C_0C_1 + C_1C_2 + C_2\,C_3 + \text{.......}$

equation of the coefficients,

$$C_0C_1 + C_1C_2 + \text{.......} = C_{n-1}\,C_n = {}^{2n}C_{n-1} = \frac{(2n)!}{(n-1)!(n+1)!}$$

Example : 8

Let $S_n = 1 + q + q^2 + q^3 + \text{.........} + q^n$

$$S_n = 1 + \left(\frac{q+1}{2}\right)^2 + \left(\frac{q+1}{2}\right)^3 + \text{.......} + \left(\frac{q+1}{2}\right)^n$$

prove that ${}^{n+1}C_1 + {}^{n+2}C_2\,S_1 + {}^{n+1}C_3\,S_2 + \text{........} + {}^{n+1}C_{n+1}\,S_n = 2^n\,S_n$

Solution

S_n = sum of (n + 1) terms of a G.P. = $\frac{1-q^{n+1}}{1-q}$

$$S_n = \frac{1-\left(\frac{q+1}{2}\right)^{n+1}}{1-\left(\frac{q+1}{2}\right)} = \frac{2^{n+1}-(q+1)^{n+1}}{(1-q)\,2^n}$$

Consider the LHS = ${}^{n+1}C_1 + {}^{n+1}C_2\left(\dfrac{1-q^2}{1-q}\right) + {}^{n+1}C_3\left(\dfrac{1-q^3}{1-q}\right) + \ldots\ldots + {}^{n+1}C_{n+1}\left(\dfrac{1-q^{n+1}}{1-q}\right)$

$$= \frac{1}{1-q}\left[{}^{n+1}C_1(1-q) + {}^{n+1}C_2(1-q^2) + \ldots\ldots\ldots\ldots + {}^{n+1}C_{n+1}(1-q^{n+1})\right]$$

$$= \frac{1}{1-q}\left[({}^{n+1}C_1 + {}^{n+1}C_2 + \ldots\ldots\ {}^{n+1}C_{n+1}) - ({}^{n+1}C_1 q + {}^{n+1}C_2 q^2 + \ldots\ldots + {}^{n+1}C_{n-1} q^{n+1})\right]$$

$$= \frac{1}{1-q}\left[(2^{n+1} - 1 - ((1+q)^{n+1} - 1)\right] = \frac{2^{n+1} - (1+q)^{n+1}}{1-q} = 2^n S_n = \text{RHS}$$

Example : 9

Show that $3^{2n+2} - 8n - 9$ is divisible by 64 if $n \in N$.

Solution

$3^{2n+2} - 8n - 9 = (1 + 8)^{n+1} - 8n - 9 = [1 + (n + 1)\,8 + ({}^{n+1}C_2\,8^2 + \ldots\ldots..] - 8n - 9$

$= {}^{n+1}C_2\,8^2 + {}^{n+1}C_3\,8^3 + {}^{n+1}C^4\,8^4 + \ldots\ldots..$

$= 64[{}^{n+1}C_2 + {}^{n+1}C_3\,8 + {}^{n+1}C_4 8^2 + \ldots\ldots.]$

which is clearly divisible by 64

Example : 10

Find numerically greatest term in the expansion of $(2 + 3x)^9$, when $x = 3/2$

Solution

$$(2 + 3x)^9 = 2^9\left(1 + \frac{3x}{2}\right)^9 = 2^9\left(1 + \frac{9}{4}\right)^9$$

Let us calculate $m = \dfrac{x(n+1)}{x+1} = \dfrac{(9/4)(9+1)}{(9/4)+1} = \dfrac{90}{13} = 6\dfrac{12}{13}$

as m is not an integer, the greatest term in the expansion is $T_{[m]+1} = T_7$

$\Rightarrow$ the greatest term $= 2^0\,(T_7) = 2^9\,{}^9C_6\left(\dfrac{9}{4}\right)^6 = \dfrac{7\times 3^{13}}{2}$

Example : 11

If a_1, a_2, a_3 and a_4 are the coefficients of any four consecutive terms in the expansion of $(1 + x)^n$, prove that

$$\frac{a_1}{a_1+a_2} + \frac{a_3}{a_3+a_4} = \frac{2a_2}{a_2+a_3}$$

Solution

Let a_1 = coefficient of $T_{r+1} = {}^nC_r \quad \Rightarrow \quad a_2 = {}^nC_{r+1} = {}^nC_r$

$\Rightarrow \quad a_2 = {}^nC_{r+1}, \quad a_3 = {}^nC_{r+2}, \quad a_4 = {}^nC_{r+3}$

$\Rightarrow \quad \dfrac{a_1}{a_1+a_2} = \dfrac{{}^nC_r}{{}^nC_r + {}^nC_{r+1}} = \dfrac{{}^nC_r}{{}^{n+1}C_{r+1}} = \dfrac{r+1}{n+1}$ and $\dfrac{a_3}{a_3+a_4} = \dfrac{{}^nC_{r+2}}{{}^nC_{r+2} + {}^nC_{r+3}} = \dfrac{{}^nC_{r+2}}{{}^{n+1}C_{r+3}} = \dfrac{r+3}{n+1}$

LHS $= \dfrac{a_1}{a_1+a_2} + \dfrac{a_3}{a_3+a_4} + \dfrac{r+1}{n+1} = \dfrac{r+3}{n+1} = \dfrac{2(r+2)}{n+1}$

RHS $= \dfrac{2a_2}{a_2+a_3} = \dfrac{2\,{}^nC_{r+1}}{{}^nC_{r+1} + {}^nC_{r+2}} = \dfrac{2\,{}^nC_{r+1}}{{}^{n+1}C_{r+2}} = \dfrac{2(r+2)}{n+1}$

Hence R.H.S. = L.H.S

Example : 12

Prove that following ($C_r = {}^nC_r$)

(a) $C_1 + 2C_2 + 3C_3 + \text{..........} \; n\,C_n = n\,2^{n-1}$

(b) $C_1 - 2C_2 + 3C_3 + - \text{.........} = 0$

(c) $C_0 + 2C_1 + 3C_2 + \text{........} + (n+1)\,C_n = (n+2)\,2^{n-1}$

Solution

Consider the identity : $(1 + x)^n = C_0 + C_1x + C_2x^2 + \text{........} + C_nx^n$

Differentiating w.r.t. x, we get another identity $n(1 + x)^{n-1}$

$$= C_1 + 2C_2\,x + 3\,C_3x^2 + \text{.........} + nC_n\,x^{n-1} \quad \text{................(i)}$$

(a) substituting x = 1 in (i), we get :

$C_1 + 2\,C_2 + 3C_3 + \text{........} + nC_n = n\,2^{n-1}$ ....(ii)

(b) Substituting x = –1 in (i), we get

$C_1 - 2C_2 + 3C_3 - 4C_4 + \text{.........} + nC_n\,(-1)^{n-1} = 0$

(c) LHS = $C_0 + 2C_1 + 3C_2 + \text{.....} + (n + 1)C_n = (C_0 + C_1 + C_2 + \text{......}) + (C_1 + 2C_2 + 3C_3 + \text{......} + nC_n)$

$= 2^n + n\,2^{n-1} = (n + 1)\,2^{n-1}$ [using (ii)]

This can also be proved by multiplying (i) by x and then differentiating w.r.t. x and then substituting x = 1.

Example : 13

Prove that

(a) $$\frac{C_0}{1} + \frac{C_1}{2} + \frac{C_2}{3} + \frac{C_3}{4} + \text{........} + \frac{C_n}{n+1} = \frac{2^{n+1}-1}{n+1}$$

(b) $$3C_0 + 3^2\,\frac{C_1}{2} + 3^3\,\frac{C_2}{3} + 3^4\,\frac{C_3}{4} + \text{......} + 3^{n+1}\,\frac{C_n}{n+1} = \frac{4^{n+1}-1}{n+1}$$

Solution

Consider the identity :

$(1 + x)^n = C_0 + C_1x + C_2\,x^2 + \text{.........} + C_nx^n$ ....(i)

(a) Integrating both sides of (i) within limits 0 to 1, we get

$$\int_0^1 (1+x)^n\,dx = \int_0^1 (C_0 + C_1x + \text{........}C_nx^n)\,dx$$

$$\left.\frac{(1+x)^{n+1}}{n+1}\right]_0^1 = \left. C_0\,x + \frac{C_1x^2}{2} + \frac{C_2x^3}{3} + \text{.........} + \frac{C_nx^{n+1}}{n+1}\right]_0^1$$

$$\frac{2^{n+1}-1}{n+1} = C_0 + \frac{C_1}{2} + \frac{C_2}{3} + \text{.........} + \frac{C_n}{n+1}$$

(b) Integrating both sides of (i) within limits – 1 to + 1, we get:

$$\int_{-1}^1 (1+x)^n\,dx = \int_{-1}^1 (C_0 + C_1x + \text{........} + C_nx^n)\,dx$$

$$\left.\frac{(1+x)^{n+1}}{n+1}\right]_{-1}^1 = \left. C_0\,x + \frac{C_1x^2}{2} + \frac{C_2x^3}{3} + \text{..........} + \frac{C_nx^{n+1}}{n+1}\right]_{-1}^{+1}$$

$$\frac{2^{n+1}-0}{n+1} = \left(C_0 + \frac{C_1}{2} - \frac{C_2}{3} + \text{.....} + \frac{C_n}{n+1}\right) - \left(-C_0 + \frac{C_1}{2} - \frac{C_2}{3} + \text{.......}\right)$$

$$\Rightarrow \quad \frac{2^{n+1}}{n+1} = 2C_0 + \frac{2C_2}{3} + \frac{2C_4}{5} + \text{..........}$$

$$\Rightarrow \quad \frac{2^n}{n+1} = C_0 + \frac{C_2}{3} + \frac{C_4}{5} + \text{.........}$$

Hence proved

Note : If the sum contains $C_0, C_1, C_2, C_3 \ldots\ldots. C_n$ (i.e. all +ve coefficients), then integrate between limits 0 to 1. If the sum contains alternate plus and minus (+ – signs), then integrate between limits – 1 to 0. If the sum contains even coefficients ($C_0, C_2, C_4 \ldots..$), then integrate between – 1 and +1.

Example : 15

$1^2\, C_1 + 2^2\, C_2 + 3^2\, C_3 + \ldots\ldots\ldots + n^2\, C_n = n(n + 1)\, 2^{n-2}$

Solution

Consider the identity :

$(1 + x)^n = C_0 + C_1x + C_2\, x^2 + \ldots\ldots\ldots\ldots + C_n\, x^n$

Differentiating both sides w.r.t. x;

$n(1 + x)^{n-1} = C_1 + 2C_2\, x + \ldots\ldots\ldots + nC_n\, x^{n-1}$

multiplying both sides by x.

$n\, x\, (1 + x)^{n-1} = C_1\, x + 2\, C_2\, x^2 + \ldots\ldots\ldots. + n\, C_n\, x^n$

differentiate again w.r.t. x;

$nx\, (n - 1)\, (1 + x)^{n-2} + n\, (1 + x)^{n-1} = C_1 + 2^2\, C_2\, x + \ldots\ldots\ldots. + n^2\, C_n\, x$

substitute x = 1 in this identity

$n(n - 1)\, 2^{n-2} + n\, 2^{n-1} = C_1 + 2^2\, C_2 + 3^2\, C_3 + \ldots\ldots\ldots\ldots + n^2\, C_n$

$\Rightarrow \quad n\, 2^{n-2}\, (n + 1) = C_1 + 2^2\, C_2 + \ldots\ldots\ldots\ldots.. + n^2\, C_n$

Hence proved

Example : 16

If ${}^{2n}C_r = C_r$, prove that : $C_1^2 - 2C_2^2 + 3C_3^2 - + \ldots\ldots\ldots\ldots - 2n\, C_{2n}^2 = (-1)^{n-1}\, nC_n$.

Solution

Consider

$(1 - x)^{2n} = C_0 - C_1\, x + C_2x^2 - + \ldots\ldots\ldots\ldots + C_{2n}\, x^{2n}$ ....(i)

and

$(x + 1)^{2n} = C_0\, x^{2n} + C_1\, x^{2n-1} + C_2\, x^{2n-2} + \ldots\ldots\ldots + C_{2n-1}\, x + C_{2n}$ ....(ii)

We will differentiate (i) w.r.t. x and then multiply with (ii)

Differentiating (i), we get :

$-\, 2n\, (1 - x)^{2n-1} = -\, C_1 + 2\, C_2\, x - 3\, C_3x^2 + \ldots\ldots\ldots\ldots + 2n\, C_{2n}\, x^{2n-1}$

$\Rightarrow \quad 2n\, (1 - x)^{2n-1} = C_1 - 2\, C_2\, x + 3\, C_3\, x^2 - + \ldots\ldots. - 2n\, C^{2n}\, x^{2n-1}$

new multiplying with (ii)

$2n\, (1 - x)^{2n-1}\, (x + 1)^{2n} = (C_0\, x^{2n} + C_1\, x^{2n-1} + \ldots\ldots. + C_{2n}) \times (C_1 - 2\, C_2\, x + 3\, C_3\, x^2 - + \ldots\ldots\ldots - 2n\, C_{2n}\, x^{2n-1})$

Comparing the coefficients of x^{2n-1} on both sides; coefficient in RHS

$= C_1^2 - 2\, C_2^2 + 3\, C_3^2 - + \ldots.. - 2n\, C_{2n}^2$

Required coeff. in LHS $=$ coeff. of x^{2n-1} in $2n\, (1 - x)^{2n-1}\, (1 + x)^{2n-1}\, (1 + x)$

$=$ coeff. of x^{2n-1} in $2n\, (1 - x^2)^{2n-1}$ + coeff. of x^{2n-1} in $2nx\, (1 - x^2)^{2n-1}$

$=$ coeff. of x^{2n-1} in $2n\, (1 - x^2)^{2n-1}$ + coeff. of x^{2n-2} in $2n\, (1 - x^2)^{2n-1}$

Now the expansion of $(1 - x^2)^{2n-1}$ contains only even powers of x.

Hence coefficients in LHS :

$= \quad 0 + 2n$ [coeff. of x^{2n-2} in $(1 - x^2)^{2n-1}$]

$= \quad 2n\, [{}^{2n-1}C_{n-1}\, (-1)^{n-1}]$

$$= \quad 2n\left(\frac{(2n-1)!}{(n-1)!\, n!}(-1)^{n-1}\right)$$

$= \quad n\, {}^{2n}C_n\, (-1)^{n-1}$

Now equating the coefficients in RHS and LHS, we get $C_1^2 - 2C_2^2 + 3C_3^2 - + \ldots\ldots.. 2n\, C_{2n}^2 = (-1)^{n-1}\, n\, {}^{2n}C_n$

Example : 17

Find the sum of series :

$$\sum_{r=0}^{n} (-1)^r\, {}^nC_r \left(\frac{1}{2^r} + \frac{3^r}{2^{2r}} + \frac{7^r}{2^{3r}} + \frac{15^r}{2^{4r}} + \ldots\ldots.m \text{ terms}\right)$$

Solution

$$\sum_{r=0}^{n}(-1)^r\ {}^nC_r\left(\frac{1}{2}\right)^r = \sum_{r=0}^{n}{}^nC_r\left(-\frac{1}{2}\right)^r = \text{expansion of}\left(1-\frac{1}{2}\right)^n$$

$$\sum_{r=0}^{n}(-1)^r\ {}^nC_r\left(\frac{3^r}{2^{2r}}\right) = \sum_{r=0}^{n}{}^nC_r\left(-\frac{3}{4}\right)^r = \text{expansion of}\left(1-\frac{3}{4}\right)^n$$

$$\sum_{r=0}^{n}(-1)^r\ {}^nC_r\left(\frac{7^r}{2^{3r}}\right) = \sum_{r=0}^{n}{}^nC_r\left(-\frac{7}{8}\right)^r = \text{expansion of}\left(1-\frac{7}{8}\right)^n \text{ and so on ....}$$

Now adding all these we get ;

$$\text{Required Sum } = \sum_{r=0}^{n}(-1)^r\ {}^nC_r\left(\frac{1}{2^r}+\frac{3^r}{2^{2r}}+\frac{7^r}{2^{3r}}+\frac{15^r}{2^{4r}}+\text{........m terms}\right)$$

$$= \left(1-\frac{1}{2}\right)^n + \left(1-\frac{3}{4}\right)^n + \left(1-\frac{7}{8}\right)^n + \text{..... m terms}$$

$$= \frac{1}{2^n} + \frac{1}{4^n} + \frac{1}{8^n} + \text{....... m terms of GP}$$

$$= \frac{\frac{1}{2^n}\left(1-\frac{1}{2^{mn}}\right)}{1-\frac{1}{2^n}} = \frac{2^{mn}-1}{(2^n-1)2^{mn}}$$

Example : 18

If $(1 + x)^n = C_0 + C_1 x + C_2x^2 + ... + C_n x^n$ then show that the sum of the products of the C_is taken two at a time represented by : $\sum\sum_{0\le i<j\le n} C_iC_j$ is equal to $2^{2n-1} - \frac{(2n)!}{2n!\,n!}$

Solution

The square of the sum of n terms is given by :

$$(C_0 + C_1 + C_2 + ... C_n)^2 = (C_0^2 + C_1^2 + C_2^2 + ... + C_n^2) + 2\sum\sum_{0\le i<j\le n} C_iC_j$$

substituting $C_0 + C_1 + C_2 + ... + C_n = 2^n$

and $C_0^2 + C_1^2 + C_2^2 + ... + C_n^2 = {}^{2n}C_n$

we get $(2^n)^2 = {}^{2n}C_n + 2\sum\sum_{0\le i<j\le n} C_iC_j \quad\Rightarrow\quad \sum\sum_{0\le i<j\le n} C_iC_j = \frac{2^{2n}-{}^{2n}C_n}{2} = 2^{2n-1} - \frac{(2n)!}{2n!\,n!}$

Example : 19

If $(2 + \sqrt{3})^n = I + f$ where I and n are positive integers and $0 < f < 1$, show that I is an odd integer and $(1 - f)(I + f) = 1$.

Solution

$(2 + \sqrt{3})^n = f'$ where $0 < f' < 1$ because $2 - \sqrt{3}$ is between 0 and 1

Adding the expansions of $(2 + \sqrt{3})^n$ and $(2 - \sqrt{3})^n$, we get ; $1 + f + f' = (2 + \sqrt{3})^n + (2 - \sqrt{3})^n$

$= 2\,[C_0\,2^n + C_2\,2^{n-2}(\sqrt{3})^2 + ...]$ = even integer ....(i)

$\Rightarrow$ $f + f'$ is also an integer

now $0 < f < 1$ and $0 < f' < 1 \quad \Rightarrow \quad 0 < f + f' < 2$

The only integer between 0 and 2 is 1

Hence $f + f' = 1$ ....(ii)

Consider (i)

$1 + f + f'$ = even integer

$\Rightarrow$ $I + 1$ = even integer [using (ii)]

$\Rightarrow$ I = odd integer

also $(I + f)(I - f) = (I + f)(f') = (2 + \sqrt{3})^n (2 - \sqrt{3})^n = 1$

Example : 20

If $(6\sqrt{6} + 14)^{2n+1} = P$, prove that the integral part of P is an even integer and $P f = 20^{2n+1}$ where f is the fractional part of P.

Solution

Let I be the integral part of P

$\Rightarrow$ $P = I + f = (6\sqrt{6} + 14)^{2n+1}$

Let $f' = (6\sqrt{6} - 14)$ lies between 0 and 1, $0 < f' < 1$

subtracting f′ from I + f to eliminate the irrational terms in RHS of (i)

$I + f - f' = (6\sqrt{6} + 14)^{2n+1} - (6\sqrt{6} - 14)^{2n+1} = 2[{}^{2n+1}C_1 \, (6\sqrt{6})^{2n} (14) + {}^{2n+1}C_3 \, (6\sqrt{6})^{2n-2} (14)^3 + ...]$

= even integer ....(ii)

$\Rightarrow$ $f - f'$ is an integer

now $0 < f < 1$ and $0 < f' < 1$

$\Rightarrow$ $0 < f < 1$ and $-1 < -f' < 0$

adding these two, we get; $-1 < f - f' < 1$

$\Rightarrow$ $f - f' = 0$ ....(iii)

Consider (ii)

$1 + f - f'$ = even integer

$\Rightarrow$ $I + 0$ = even integer [using (iii)]

$\Rightarrow$ integral part of P is even

Also $P f = (I + f) f = (1 + f) f' = (6\sqrt{6} + 14)^{2n+1} (6\sqrt{6} - 14)^{2n+1} = 216 - 196)^{2n+1} = 20^{2n+1}$

Example : 21

Expand $\dfrac{2 - x}{(1 - x)(3 - x)}$ in ascending powers of x and find x^r. Also state the range of x for which this expression is valid.

Solution

Given expression = $\dfrac{2 - x}{(1 - x)(3 - x)}$

On expressing RHS in the form of partial fractions, we get

Given expression = $\dfrac{1}{2(1 - x)} + \dfrac{1}{2(3 - x)}$

$\Rightarrow$ Given expression = $\dfrac{1}{2}(1 - x)^{-1} + \dfrac{1}{6}\left(1 - \dfrac{x}{3}\right)^{-1}$

Using the expansions of $(1 - x)^{-1}$, we get

Given expression = $\dfrac{1}{2}(1 + x + x^2 + x^3 + ...) + \dfrac{1}{6}\left(1 + \dfrac{x}{3} + \dfrac{x^2}{9} + \dfrac{x^3}{27} + ...\right)$

$\Rightarrow$ Given expansion = $\left(\dfrac{1}{2} + \dfrac{1}{6}\right) + \left(\dfrac{1}{2} + \dfrac{1}{18}\right) x + \left(\dfrac{1}{2} + \dfrac{1}{54}\right) x^2 + ... + \left(\dfrac{1}{2} + \dfrac{1}{63^r}\right) x^r + ...$

$\Rightarrow$ Given expression = $\dfrac{2}{3} + \dfrac{5}{9} x + \dfrac{14}{27} x^2 + ... + \dfrac{1}{2}\left(1 + \dfrac{1}{3^{r+1}}\right) x^r + ...$

Coefficient of $x^r = \dfrac{1}{2}\left(1 + \dfrac{1}{3^{r+1}}\right) x^r$

Since $(1 - x)^{-1}$ is valid for $x \in (-1, 1)$ and $(1 - x/3)^{-1}$ is valid for $x \in (-3, 3)$, the given expression is valid for $x \in (-1, 1)$ (i.e. take intersection of the two sets)

Hence $\frac{2-x}{(1-x)(3-x)}$ is valid for $-1 < x < 1$

Example : 22

If $y = \frac{3}{4} + \frac{3.5}{4.8} + \frac{3.57}{4.812} + \dots\dots\dots$ till infinity, show that $y^2 + 2y - 7 = 0$

Solution

It is given that : $y = \frac{3}{4} + \frac{3.5}{4.8} + \frac{3.57}{4.812} + \dots\dots$ to ∞

On adding 1 to both sides, we get :

$1 + y = 1 + \frac{3}{4} + \frac{3.5}{4.8} + \frac{3.57}{4.812} + \dots\dots$ to ∞ ....(i)

Now we will find the sum of series on RHS or (i)

For this consider the expansion of $(1 + t)^n$, where n is negative or fraction :

$(1 + t)^n = 1 + nt + \frac{n(n-1)}{1.2} t^2 + \frac{n(n-1)(n-2)}{1.2.3} t^3 + \dots\dots$ to ∞ where $|t| < 1$ ....(ii)

On comparing (i) and (ii), we get

$nt = 3/4$ ....(iii)

$\frac{n(n-1)}{1.2} t^2 = \frac{3.5}{4.8}$ ....(iv)

and $(1 + t)^n = 1 + y$

Consider (iv) : $\frac{n(n-1)}{1.2} t^2 = \frac{3.5}{4.8}$

$\Rightarrow \quad \frac{(n-1)t}{2} = \frac{5}{8}$ [using (iii)]

$\Rightarrow \quad (n - 1)\, t = \frac{5}{4}$

$\Rightarrow \quad nt - t = \frac{5}{4}$

$\Rightarrow \quad \frac{3}{4} - t = \frac{5}{4}$ [using (iii)]

$\Rightarrow \quad t = -1/2$ and $n = -3/2$

$\Rightarrow$ Sum of series on RHS of (i) = $\left(1-\frac{1}{2}\right)^{-3/2}$

$\Rightarrow \quad 1 + y = (1 - 1/2)^{-3/2} \quad \Rightarrow \quad 2^{3/2} = 1 + y$

On squaring, we get $8 = (1 + y)^2$

$\Rightarrow \quad y^2 + 2y - 7 = 0$

Hence proved

Example : 23

Find the coefficient of $x_1^2\, x_2\, x_3$ in the expansion of $(x_1 + x_2 + x_3)^4$.

Solution

To find the required coefficient, we can use multinomial theorem in the question.

The coefficient of $x_1^2 x_2 x_3$ in the expansion of $(x_1 + x_2 + x_3)^4 = \frac{4!}{2!\,1!\,1!} = 12$

Hence coefficient of $x_1^2 x_2 x_3 = 12$
Note : Also try to solve this question without the use of multinomial theorem

Example : 24

Find the coefficient of x^7 in the expansion of $(1 + 3x - 2x^3)^{10}$.

Solution

Using the multinomial theorem, the general term of the expansion is :

$$T_{p,q,r} = \frac{10!}{p!\,q!\,r!}(1)^p (3x)^q (-2x^3)^r,$$

where $p + q + r = 10$. Find the coefficient of x^7 , we must have $q + 3r = 7$.

<u>Consider q + 3r = 7</u>

From the above relationship, we can find the possible values which p, q and r can take
Take $r = 0$
$\Rightarrow$ $q = 7$ and $p = 3$
$\Rightarrow$ $(p, q, r) \equiv (3, 7, 0)$ ....(i)
Take $r = 1$
$\Rightarrow$ $q = 4$ and $p = 5$
$\Rightarrow$ $(p, q, r) \equiv (5, 4, 1)$ ....(ii)
Take $r = 2$
$\Rightarrow$ $q = 1$ and $p = 7$
$\Rightarrow$ $(p, q, r) \equiv (7, 1, 2)$ ....(iii)
If we take $r > 2$, we get $q < 0$, which is not possible.
Hence (i), (ii) and (iii) and the only possible combination of values which p, q and r can take.

Using (i), (ii) and (iii), coefficient of $x^7 = \frac{10!}{1!\,3!\,7!} 3^7 + \frac{10!}{5!\,4!\,1!} 3^4 (-2)^1 + \frac{10!}{7!\,2!\,1!} 3^1 (-2)^2 = 62640$

Hence coefficient of $x^7 = 62640$

Example : 25

Find the coefficient of x^{50} in the expansion : $(1 + x)^{1000} + 2x (1 + x)^{999} + 3x^2 (1 + x)^{998} + ... + 1001x^{1000}$.

Solution

It can be easily observed that series is an Arithmetic-Geometric series with common difference = 1, common ratio = x/(1+x) and number of terms = 1001
Let $S = (1 + x)^{1000} + 2x (1 + x)^{999} + 3x^2 (1 + x)^{998} + ... + 1001x^{1000}$ ....(i)
Multiple both sides by x/(1 + x) to get
$xS/(1 + x) = x (1 + x)^{999} + 2x^2 (1 + x)^{998} + 3x^3 (1 + x)^{997} + ... 1000x^{1000} + 1001x^{1001}/(1 + x)$ ....(ii)
Shift (ii) by one term and subtract it from (i) to get :
$S/(1 + x) = (1 + x)^{1000} + x (1 + x)^{999} + x^2 (1 + x)^{998} + ... x^{1000} - 1001x^{1001} /(1 + x)$
$\Rightarrow$ $S = (1 + x)^{1001} + x (1 + x)^{1000} + x^2 (1 + x)^{999} + ... x^{1000} (1 + x) - 1001 x^{1001}$
Now the above series, upto the term $x^{1000} (1 + x)$, is G.P. with first term $= (1 + x)^{1001}$, common ratio = x/(1 + x) and number of terms = 1001

$$\Rightarrow \quad S = \frac{(1+x)^{1001}\left[1-\left(\frac{x}{1+x}\right)^{1001}\right]}{1-\frac{x}{1+x}} - 1001\,x^{1001}$$

$\Rightarrow$ $S = (1 + x)^{1002} - x^{1001} (1 + x) - 1001x^{1001}$
Coefficient of x^{50} in the series S = coeff. of x^{50} in $(1 + x)^{1002}$ ($\because$ other terms can not produce x^{50})
$\Rightarrow$ Coefficient of x^{50} in the series S = ${}^{1002}C_{50}$
Hence the coefficient of x^{50} in the given series = ${}^{1002}C_{50}$

Example : 26

Find the total number of terms in the expansion of $(x + y + z + w)^n$, $n \in N$.

Solution

Consider the expansion :

$(x + y + z + w)^n = (x + y)^n + {}^nC_1 (x + y)^{n-1} (z + w) + {}^nC_2 (x + y)^{n-2} (z + w)^2 + ... + {}^nC_n (z + w)^n$

Number of terms on the RHS = $(n + 1) + n.2 + (n - 1) . 3 + ... + (n + 1)$

$$= \sum_{r=0}^{n} (n - r + 1)(r + 1) = \sum_{r=0}^{n} (n+1) + \sum_{r=0}^{n} nr - \sum_{r=0}^{n} r^2$$

$$= (n + 1) \sum_{r=0}^{n} 1 + n\sum_{r=0}^{n} r - \sum_{r=0}^{n} r^2 = (n + 1)(n + 1) + \frac{n(n)(n+1)}{2} - \frac{n(n+1)(2n+1)}{6}$$

$$= \frac{(n+1)}{6} [6(n + 1) + 3n^2 - 2n^2 - n] = \frac{n+1}{6} [n^2 + 5n + 6] = \frac{(n+1)(n+2)(n+3)}{6}$$

Using multinomial theorem :

$(x + y + z + w)^n = \sum_{r=0}^{n} \frac{n!\, x^{n_1} y^{n_2} z^{n_3} w^{n_4}}{n_1!\, n_2!\, n_3!\, n_4!}$, where n_1, n_2, n_3 and n_4 can have all possible values for 0, 1, 2, ..., n subjected to the condition $n_1 + n_2 + n_3 + n_4 = n$ ....(i)

Therefore, the number of distinct terms in the multinomial expansion is same as the non-negative integral solutions of (i)

$\Rightarrow$ Number of distinct terms = Number of non-negative integral solutions

$\Rightarrow$ Number of distinct terms = coefficient of x^n in the expansion $(1 + x + x^2 + ...+ x^n)^4$

$$= \text{coefficient of } x^n \text{ in } \left(\frac{1 - x^{n+1}}{1 - x}\right)^4$$

$$= \text{coefficient of } x^n \text{ in } (1 - x^{n+1})^4 (1 - x)^{-4} = {}^{n+4-1}C_{4-1} = {}^{n+3}C_3$$

$\Rightarrow$ Number of distinct terms = $\frac{(n+1)(n+2)(n+3)}{6}$

Example : 27

Let n be a positive integer and $(1 + x + x^2)^n = a_0 + a_1x + a_2x^2 + ... + a_{2n} x^{2n}$.

Show that $a_0^2 - a_1^2 - + ... + a_{2n}^2 = 2_n$.

Solution

Consider the given identity : $(1 + x + x^2)^n = a_0 + a_1x + a_2x^2 + ... + a_{2n} x^{2n}$ ....(i)

Replace x by $-1/x$ in this identity to get :

$$\left(1 - \frac{1}{x} + \frac{1}{x^2}\right)^n = a_0 - \frac{a_1}{x} \frac{a_2}{x^2} - + ... + \frac{a_{2n}}{x^{2n}}$$

$\Rightarrow$ $(1 - x + x^2)^n = a_0 x^{2n} - a_1 x^{2n-1} + a_2 x^{2n-2} - + ... + a_{2n}$ ....(ii)

Multiply (i) and (ii) and also compare coefficient of x^{2n} on both sides to get :

$a_0^2 - a_1^2 + a_2^2 - + ... + a_{2n}^2$ = coefficient of x^{2n} in $(1 + x + x^2)^n (1 - x + x^2)^n$

$\Rightarrow$ LHS = coefficient of x^{2n} in $(1 + x^2 + x^4)^n$

$\Rightarrow$ LHS = coefficient of x^{2n} in $a_0 + a_1x^2 + a_2x^4 + ... + a_nx^{2n} + ... + a_{2n}x^{4n}$ [replace x by x^2 in (i)]

$\Rightarrow$ LHS = a_n

Hence $a_0^2 - a_1^2 + a_2^2 - +........ + a_{2n}^2 = a_n$

Example : 28

If $\sum_{r=0}^{2n} a_r(x-2)^r = \sum_{r=0}^{2n} b_r(x-3)^r$ and $a_k = 1$ for all $k \geq n$, show that $b_n = {}^{2n+1}C_{n+1}$.

Solution

Let $y = x - 3 \quad \Rightarrow \quad y + 1 = x - 2$

So given expression reduces to :

$$\sum_{r=0}^{2n} a_r(y+1)^r = \sum_{r=0}^{2n} b_r(y)^r$$

$\Rightarrow \quad a_0 + a_1(y + 1) + \ldots\ldots\ldots + a_{2n}(y + 1)^{2n} = b_0 + b_1y + \ldots\ldots\ldots + b_{2n}y^{2n}$

Using $a_k = 1$ for all $k \geq n$, we get

$\Rightarrow \quad a_0 + a_1(y + 1) + \ldots\ldots\ldots\ldots\ a_{n-1}(y + 1)^{n-1} + (y + 1)^n + \ldots\ldots\ldots\ldots + (y + 1)^{2n}$

$= b_0 + b_1y + \ldots\ldots. + b_ny^n + \ldots\ldots..+ b_{2n}y^{2n}$

Compare coefficient of y^n on both sides, we get :

${}^nC_n + {}^{n+1}C_n + {}^{n+2}C_n + \ldots\ldots\ldots. + {}^{2n}C_n = b_n$

Using the formula, ${}^nC_r = {}^nC_{n-r}$, we get :

${}^nC_0 + {}^{n+1}C_1 + {}^{n+2}C_2 + \ldots\ldots\ldots + {}^{2n}C_n = b_n$

Using, ${}^nC_0 = {}^{n+1}C_0$ for first term, we get :

${}^{n+1}C_0 + {}^{n+1}C_1 + {}^{n+2}C_2 + \ldots\ldots\ldots. + {}^{2n}C_n = b_n$

On combining the first two terms with use of the formula,

${}^nC_{r-1} + {}^nC_r = {}^{n+1}C_r$, we get :

${}^{n+2}C_1 + {}^{n+2}C_2 + \ldots\ldots.. + {}^{2n}C_n = b_n$

If we combine terms on LHS like we have done in last step, finally we get :

${}^{2n}C_n = b_n \quad \Rightarrow \quad b_n = {}^{2n+1}C_{n+1} \quad$ (using ${}^nC_r = {}^nC_{n-r}$)

Hence $b_n = {}^{2n+1}C_{n+1}$

Example : 29

Prove that $\sum_{r=1}^{k} (-3)^{r-1}\ {}^{3n}C_{2r-1} = 0$, where $k = 3n/2$ and n is an even positive integer.

Solution

Let $n = 2m \quad \Rightarrow \quad k = 3m$

LHS = $\sum_{r=1}^{2m} (-3)^{r-1}\ {}^{6m}C_{2r-1} = {}^{6m}C_1 - 3\ {}^{6m}C_3 + 9\ {}^{6m}C_5 - \ldots\ldots. + (-3)^{3m-1}\ {}^{6m}C_{6m-1}$ ....(i)

Consider $\quad (1 + x)^{6m} = {}^{6m}C_0 + {}^{6m}C_1x + {}^{6m}C_2\ x^2 + \ldots\ldots\ldots. + {}^{6m}C_{6m}\ x^{6m}$ and

$(1 - x)^{6m} = {}^{6m}C_0 - {}^{6m}C_1x + {}^{6m}C_2\ x^2 + \ldots\ldots\ldots.+ {}^{6m}C_{6m}\ x^{6m}$

On subtracting the above two relationships, we get

$(1 + x)^{6m} - (1 - x)^{6m} = 2\ ({}^{6m}C_1\ x + {}^{6m}C_3x^3 + {}^{6m}C_5x^5 + \ldots\ldots\ldots + {}^{6m}C_{6m-1}\ x^{6m-1})$

Divide both sides by 2x to get :

$$\frac{(1+x)^{6m} - (1-x)^{6m}}{2x} = {}^{6m}C_1 + {}^{6m}C_3\ x^2 + \ldots\ldots.. + {}^{6m}C_{6m-1}\ x^{6m-2}$$

Put $x = \sqrt{3}i$ in the above identity to get :

$$\frac{(1+i\sqrt{3})^{6m} - (1-i\sqrt{3})^{6m}}{2\sqrt{3}i} = {}^{6m}C_1 - 3\ {}^{6m}C_3 + \ldots\ldots\ldots. + (-3)^{3m-1}\ {}^{6m}C_{6m-1} \qquad \text{...........(ii)}$$

Comparing (i) and (ii), we get

$$\text{LHS} = \frac{2^{6m}\left[\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)^{6m} - \left(\cos\frac{\pi}{3} - i\sin\frac{\pi}{3}\right)^{6m}\right]}{2\sqrt{3}i}$$

$\Rightarrow$ LHS = $\dfrac{2^{6m}[(\cos 2\pi m + i\sin 2\pi m) - (\cos 2\pi m - i\sin 2\pi m)]}{2\sqrt{3}i}$ (using De morvie's Law)

$\Rightarrow$ LHS = $\dfrac{2^{6m}2i\sin 2\pi m}{2\sqrt{3}i} = \dfrac{2^{6m}\sin 2\pi m}{\sqrt{3}} = 0$ (because $\sin 2\pi m = 0$)

Example : 30

Show by expanding $[(1 + x)^n - 1]^m$ where m and n are positive integers, that

$^mC_1\, {}^nC_m - {}^mC_2\, {}^{2n}C_m + {}^mC_3\, {}^{3n}C_m$ .... $= (-1)^{m-1} n^m$.

Solution

Consider : $[(1 + x)^n - 1]^m$ and expand $(1 + x)^n$ binomially

$\Rightarrow$ $[(1 + x)^n - 1]^m = [1 + {}^nC_1x^2 + \ldots\ldots\ldots + {}^nC_n\, x^n) - 1]^m$

$\Rightarrow$ $[(1 + x)^n - 1]^m = [{}^nC_1\, x + {}^nC_2\, x^2 + \ldots\ldots. + {}^nC_n\, x^n]^m$

$\Rightarrow$ $[(1 + x)^n - 1]^m = x^m\, [{}^nC_1 + {}^nC_2\, x + \ldots\ldots..+ {}^nC_n\, x^{n-1}]^m$ ....(i)

Now consider : $[(1 + x)^n - 1]^m = (-1)^m\, [1 - (1 + x)^n]^m$

$\Rightarrow$ $[(1 + x)^n - 1)^m = (-1)^m\, [1 - {}^mC_1\, (1 + x)^n + {}^mC_2\, (1 + x)^{2n} - \ldots\ldots\ldots\ldots..]$ ....(ii)

Comparing (i) and (ii), we get :

$x^m\, [{}^nC_1 + {}^nC_2x + \ldots\ldots\ldots + {}^nC_nx^{n-1}]^m\, [1 - {}^mC_1\, (1 + x)^n + {}^mC_2\, (1 + x)^{2n} - \ldots\ldots..]$

Compare coefficient of x^m on both sides to get :

$n^m = (-1)^m\, [-{}^mC_1\ {}^nC_m + {}^mC_2\ {}^{2n}C_m - {}^mC_3\ {}^{3n}C_m + \ldots\ldots\ldots.]$

$\Rightarrow$ ${}^mC_1\, {}^nC_m - {}^mC_2\, {}^{2n}C_m + {}^mC_3\, {}^{3n}C_m - + \ldots\ldots\ldots = (-1)^{m-1}\, n^m$

Hence proved

Example : 31

Show that $\displaystyle\sum_{r=1}^{n} (-1)^{r-1}\frac{C_r}{r} = \sum_{r=1}^{n}\frac{1}{r}$

Solution

Consider : $(1 - x)^n = C_0 - C_1x + C_2x^2 - \ldots\ldots\ldots.. + (-1)^n\, C_n\, x^n$

$\Rightarrow$ $1 - (1 - x)^n = C_1x - C_2\, x^2 + C_3\, x^3 + \ldots\ldots.. + (-1)^{n-1}\, C_nx^n$ ($\because$ $C_0 = 1$)

Divide both sides by x to get :

$\dfrac{1-(1-x)^n}{x} = C_1 - C_2\, x + C_3x^2 + \ldots\ldots\ldots + (-1)^{n-1}\, C_n\, x^{n-1}$

Integrate both sides between limits 0 and 1 to get :

$$\int_0^1 \frac{1-(1-x)^n}{x} = \int_0^1 [C_1 - C_2x + C_3x^2 + \ldots\ldots + (-1)^{n-1}C_nx^{n-1}]\, dx$$

$$\Rightarrow \int_0^1 \frac{1-(1-x)^n}{1-(1-x)} = C_1x - C_2\,\frac{x^2}{2} = C_3\,\frac{x^3}{3} - \ldots\ldots\ldots + (-1)^{n-1}C_n\,\frac{x^n}{n}\Bigg]_0^1$$

It can be easily observed that integrand on the LHS is the summation of n terms of G.P. whose first term is 1 and common ratio is (1 – x).

$$\Rightarrow \int_0^1 [1+(1-x)+(1-x)^2 + \ldots\ldots\ldots + (1-x)^{n-1}]\, dx = C_1 - \frac{C_2}{2} + \frac{C_3}{3} - + \ldots\ldots.. + \frac{(-1)^{n-1}C_n}{n}$$

$$\Rightarrow x - \frac{(1-x)^2}{2} - \frac{(1-x)^3}{3} - \ldots\ldots\ldots.. - \frac{(1-x)^n}{n}\Bigg]_0^1 = C_1 - \frac{C_2}{2} + \frac{C_3}{3} - \ldots\ldots. + \frac{(-1)^{n-1}C_n}{n}$$

$$\Rightarrow 1 + \frac{1}{2} = \frac{1}{3} + \ldots\ldots\ldots.. + \frac{1}{n} = C_1 - \frac{C_2}{2} + \frac{C_3}{3} - \ldots\ldots\ldots = \frac{(-1)^{n-1}C_n}{n}$$

$$\Rightarrow \quad \sum_{r=1}^{n} (-1)^{r-1} \frac{C_r}{r} = \sum_{r=1}^{n} \frac{1}{r}$$. Hence proved

Example : 32

Show that $\frac{C_0}{1} - \frac{C_1}{5} + \frac{C_2}{9} - \frac{C_3}{13} + \ldots\ldots\ldots + (-1)^n \frac{C_n}{4n+1} = \frac{4^n n!}{1.5.9......(4n+1)}$

Solution

On observing the LHS of the relationship to be proved, we can conclude that the expansion of $(1 - x^4)^n$ must be used to prove LHS equals RHS Hence,

$(1 - x^4)^n = C_0 - C_1x^4 + C_2x^8 - C_3x^{12} + \ldots\ldots\ldots + (-1)\ {}^nC_n\ x^{4n}$

Integrating both sides between limits 0 and 1, we get :

$$\int_0^1 (1-x^4)^n = \frac{C_0}{1} - \frac{C_1}{5} + \frac{C_9}{9} - \frac{C_{13}}{13} + \ldots\ldots\ldots + (-1)^n \frac{C_n}{4n+1} \qquad \ldots\ldots\ldots(i)$$

Let $\quad I_n = \int_0^1 (1-x^4)^n\ dx \qquad \ldots\ldots\ldots\ (ii)$

apply by-parts taking $(1 - x^4)^n$ as the I part and dx as the II part ,

$$\Rightarrow \quad I_n = (1 - x^4)^n\ x]_0^1 - \int_0^1 n\,(1-x^4)^{n-1}\ (-4x^3)\ x\ dx$$

$$\Rightarrow \quad I_n = 4n \int_0^1 x^4(1-x^4)^{n-1}dx = 4n \int_0^1 [1-(1-x^4)](1-x^4)^{n-1}\ dx$$

$$\Rightarrow \quad I_n = 4n \int_0^1 (1-x^4)^{n-1}\ dx - 4n \int_0^1 (1-x^4)^n\ dx$$

$$\Rightarrow \quad I_n = 4n\ I_{n-1} - 4n\ I_n$$

$$\Rightarrow \quad I_n = \frac{4n}{4n+1}\ I_{n-1}$$

Replace n by 1, 2, 3, 4, ..., n–1 in the above identity and multiply all the obtained relations,

$$\Rightarrow \quad I_n = \frac{4n}{4n+1} \cdot \frac{4(n-1)}{4n-3} \cdot \frac{4(n-2)}{4n-7} \ldots\ldots\ldots\ \frac{4}{5}\ I_0 \qquad \ldots\ldots\ldots.(iii)$$

Finding I_0

I_0 can be obtained by substituting n = 0 in (ii) i.e.

$$I_0 = \int_0^1 (1-x^4)^0\ dx = \int_0^1 dx = 1$$

Substitute the value of I_0 in (iii) to get :

$$I_n = \frac{4n}{4n+1} \cdot \frac{4(n-1)}{4n-3} \cdot \frac{4(n-2)}{4n-7} \ldots\ldots\ldots\ \frac{4}{5}$$

$$\Rightarrow \quad I_n = \frac{4^n\ n!}{1.5.9.13.......(4n+1)}$$

Using (i)

$$\frac{C_0}{1} - \frac{C_1}{5} + \frac{C_2}{9} - \frac{C_3}{13} + \ldots\ldots + (-1)^n \frac{C_n}{4n+1} = \frac{4^n\, n!}{1.5.9.....(4n+1)}$$

Hence proved

Example : 33

Show that $x^n - y^n$ is divisible by $x - y$ if n is natural number.

Solution

Let $P(n) = x^n - y^n$ is divisible by $x - y$

We consider P(1)

P(1) : $x^1 - y^1$ is divisible by $x - y$

$\Rightarrow$ P(1) is true

Now let us assume p(k) to be true

i.e. we are given P(k) : $x^k - y^k$ is divisible by $x - y$

Let $x^k - y^k = (x - y)\, m,\ m \in I$

Consider P(k + 1) :

P(k + 1) : $x^{k+1} - y^{k+1}$ is divisible by $x - y$;

$$\begin{aligned}\text{Now } x^{k+1} - y^{k+1} &= x^{k+1} - x^k y + x^k y - y^{k+1} \\ &= x^k (x - y) + y\,(x^k - y^k) \\ &= x^k (x - y) + y\,(x - y)m \\ &= (x - y)\,(x^k + my)\end{aligned}$$

Hence P(k + 1) is true whenever P(k) is true.

Hence according to the principle of Mathematical Induction, P(n) is true for all natural numbers.

Example : 34

Show that $5^{2n+2} - 24n - 25$ is divisible by 576.

Solution

Let P(n) : $5^{2n+2} - 24 - 25$ is divisible by 576

P(1) : $5^{2(1)+2} - 24\,(1) - 25$ is divisible by 576

P(1) : 576 is divisible by 576

$\Rightarrow$ P(1) is true

P(k) : $5^{2k+2} - 24k - 25 = 576m,\ m \in N$

P(k + 1) : $5^{2k+4} - 24\,(k + 1) - 25$ is divisible by 576

Consider $5^{2k+4} - 24\,(k + 1) - 25$

$$\begin{aligned}&= 5^{2k+4} - 24\,(k + 1) - 25 \\ &= 5^{2k+2} \,.\, 5^2 - 24k - 49 \\ &= 25\,(24k + 25 + 576m) - 24k - 49 \qquad \text{[using P(k)]} \\ &= (576)\,25m - 576k + 576 \\ &= 576\,(25m - k + 1)\end{aligned}$$

$\Rightarrow$ $5^{2k+4} - 24\,(k + 1) - 25$ is divisible by 576

Hence P(k + 1) is true whenever p(k) is true

Hence according to the principle of Mathematical Induction P(n) is true for all natural numbers.

Example : 35

Show that $2^n > n$ for all natural numbers

Solution

Let P(n) : $2^n > n$

P(1) : $2^1 > 1$

$\Rightarrow$ P(1) is true

P(k) : $2^k > k$

Assume that p(k) is true

P(k + 1) : $2^{k+1} > k + 1$

consider P(k) : $2^k > k$

$\Rightarrow$ $2^{k+1} > 2k$

$\Rightarrow$ $2^{k+1} > k + k$

But we have $k \geq 1$

Adding $2^{k+1} + k > k + k + 1$

$2^{k+1} > k + 1$

Hence P(k + 1) is true whenever P(k) is true

Hence according to the principle of Mathematical Induction, P(n) is true for all natural numbers.

Example : 36

Prove by the method of Induction that : $\frac{1}{3.7} + \frac{1}{7.11} + \frac{1}{11.15} + ... + \frac{1}{(4n-1)(4n+3)} = \frac{n}{3(4n+3)}$

Solution

Let P(n) : $\frac{1}{3.7} + \frac{1}{7.11} + \frac{1}{11.15} + ... + \frac{1}{(4n-1)(4n+3)} = \frac{n}{3(4n+3)}$

P(1) : $\frac{1}{3.7} = \frac{1}{3(4+3)}$

P(1) : $\frac{1}{21} = \frac{1}{21}$

$\Rightarrow$ P(1) is true

P(k) $\frac{1}{3.7} + \frac{1}{7.11} + ... + \frac{1}{(4k-1)(4k+3)} = \frac{k}{3(4k+3)}$

Assume that P(k) is true

P(k + 1) : $\frac{1}{3.7} + \frac{1}{7.11} + ... + \frac{1}{(4k-1)(4k+3)} + \frac{1}{(4k+3)(4k+7)} = \frac{k+1}{3(4k+7)}$

$$\text{LHS} = \left(\frac{1}{3.7} + \frac{1}{7.11} + ... + k \text{ terms}\right) + \frac{1}{(4k+3)(4k+7)}$$

$$= \frac{k}{3(4k+3)} + \frac{1}{(4k+3)(4k+7)} \qquad \text{[using P(k)]}$$

$$= \frac{k(4k+7)+3}{3(4k+3)(4k+7)}$$

$$= \frac{(4k+3)(k+1)}{3(4k+3)(4k+7)} \qquad = \frac{(k+1)}{3(4k+7)} = \text{RHS of P(k + 1)}$$

Hence P(k + 1) is true whenever P(k) is true

Hence according to the principle of Mathematical Induction, P(n) is true for all natural numbers.

Example : 37

Using Mathematical Induction, show that $n(n^2 - 1)$ is divisible by 24 if n is an odd positive integer.

Solution

To prove a statement for odd numbers only, it is required to show that

(a) P(1) is true

(b) P(k + 2) is true whenever p(k) is true

P(1) : $1 (1^2 - 1)$ is divisible by 24

$\Rightarrow$ P(1) is true

P(k) : k(k2 – 1) is divisible by 24 if k is odd

Assume that P(k) is true

Let $k (k^2 - 1) = 24m$ where $m \in N$

P(k + 2) : $(k + 2) [(k + 2)^2 - 1]$ is divisible by 24, if k is odd

Consider $(k + 2) [(k + 2)^2 - 1]$

$= (k + 2) (k^2 + 4k + 3)$

$= k^3 + 6k^2 + 11k + 6$
$= (24m + k) + 6k^2 + 11k + 6$
$= (24m + 6k^2 + 12k + 6$ [using P(k)]
$= 24m + 6(k + 1)^2$
$= 24m + 6(2p)^2$ [$\because$ k is odd]
$= 24m + 24p^2$
$= 24(m + p^2)$

Hence P(k + 2) is true whenever P(k) is true

Hence according to the principle of Mathematical Induction, P(n) is true for all natural numbers.

Example : 38

Prove that cos x cos 2x cos 4x .... $\cos 2^{n-1} x = \frac{\sin 2^n x}{2^n \sin x}$

Solution

P(1) : $\cos x = \frac{\sin 2x}{2 \sin x}$

P(1) : cos x = cos x (using sin 2x = 2 sin x cos x)

$\Rightarrow$ P(1) is true

P(k) : cos x cos 2x cos 4x .... $\cos 2^{k-1} x = \frac{\sin 2^k x}{2^k \sin x}$

Let P(k) be true. Consider P(k + 1)

P(k = 1) : cos x cos 2x cos 4x .... $\cos 2^{k-1} x \cos 2^k x = \frac{\sin 2^{k+1} x}{2^{k+1} \sin x}$

$$\text{LHS} = \left(\frac{\sin 2^k x}{2^k \sin x}\right) \cos 2^k x = \frac{2 \sin 2^k x \cos 2^k x}{2^{k+1} \sin x} = \frac{\sin 2^{k+1} x}{2^{k+1} \sin x} = \text{RHS}$$

Hence P(k + 1) is true whenever P(k) is true

$\therefore$ by mathematical induction P(n) is true $\forall$ n $\in$ N

Example : 39

By the method of induction, show that $(1 + x)^n \geq 1 + nx$ for n N, $x > -1$, $x \neq 0$

Solution

Let P(n) : $(1 + x)^n \geq 1 + nx$

$\Rightarrow$ P(1) : $(1 + x)^1 \geq 1 + x$ which is true

Let P(k) be true $\Rightarrow$ $(1 + x)^k \geq 1 + kx$ ....(i)

Consider P(k + 1) : $(1 + x)^{k+1} \geq 1 + (k + 1)x$

From (i) : $(1 + x)^k \geq 1 + kx$

$\Rightarrow$ $(1 + x)^{k+1} \geq (1 + kx)(1 + x)$ (as $(1 + x) > 0$)

$\Rightarrow$ $(1 + x)^{k+1} \geq 1 + (k + 1)x + kx^2$

as kx^2 is positive, it can be removed form the smaller side.

$\Rightarrow$ $(1 + x)^{k+1} \geq 1 + (k + 1)x$

$\Rightarrow$ P(k + 1) is true

Hence P(1) is true and P(k + 1) is true whenever P(k) is true

$\Rightarrow$ By induction, P(n) is true for all n $\in$ N

Example : 40

Prove that $x(x^{n-1} - na^{n-1}) + a^n (n - 1)$ is divisible by $(x - a)^2$ for n > 1 and n $\in$ N

Solution

Let P(n) : $x(x^{n-1} - na^{n-1}) + a^n (n - 1)$ is divisible by $(x - a)^2$

As n > 1, we will start from P(2)

For n = 2, the expression becomes

$= x(x - 2a) + a^2 (2 - 1) = (x - a)^2$ which is divisible by $(x - a)^2$

$\Rightarrow$ P(2) is true

Let P(k) be true

$\Rightarrow$ $x(x^{k-1} - ka^{k-1}) + a^k(k - 1)$ is divisible by $(x - a)^2$

For n = k + 1, the expression becomes $= x[x^k - (k + 1)a^k] + a^{k+1}k = x^{k+1} - kxa^k - xa^k + ka^{k+1}$

$= [x^{k+1} - kx^2a^{k-1} + xa^k(k - 1)] + kx^2a^{k-1} - xa^k(k - 1) - kxa^k - xa^k + ka^{k+1}$

$= x[x(x^{k-1} - ka^{k-1}) + a^k(k - 1)] + ka^{k-1}(x^2 - 2ax + a^2)$

= divisible by $(x - a)^2$ from P(k) + $ka^{k-1}(x - a)^2$

Hence the complete expression is divisible by $(x - a)^2$

$\Rightarrow$ P(K + 1) is true

Hence P(2) is true and P(k + 1) is true whenever P(k) is true

$\Rightarrow$ By induction, P(n) is true for all $n > 1$, $n \in N$

Alternate Method : Let $f(x) = x(x^{n-1} - na^{n-1}) + a^n(n - 1)$

It can be show that $f(a) = f'(a) = 0$

$\Rightarrow$ f(x) is divisible by $(x - a)^2$

Example : 41

For any natural number n > 1, prove that $\frac{1}{n+1} + \frac{1}{n+2} + \dots\dots\dots + \frac{1}{2n} > \frac{13}{24}$

Solution

Let P(n) : $\frac{1}{n+1} + \frac{1}{n+2} + \dots\dots\dots + \frac{1}{2n} > \frac{13}{24}$

for n = 2, $\frac{1}{2+1} + \frac{1}{2+2} > \frac{13}{24} \Rightarrow \frac{7}{12} > \frac{13}{24}$ which is true

$\Rightarrow$ P(2) is true

Let P(k) be true

$$\Rightarrow \quad \frac{1}{k+1} + \frac{1}{k+2} + \dots\dots\dots\dots + \frac{1}{2k} > \frac{13}{24}$$

Consider P(k + 1) :

$$\Rightarrow \quad \frac{1}{k+2} + \frac{1}{k+3} + \dots\dots\dots + \frac{1}{(k+1)+(k+1)} > \frac{13}{24}$$

Using P(k) we have :

$$\Rightarrow \quad \frac{1}{k+1} + \frac{1}{k+2} + \dots\dots\dots + \frac{1}{2k} > \frac{13}{24}$$

adding $\frac{1}{2k+1} + \frac{1}{2k+2} - \frac{1}{k+1}$ on both sides, we get

$$\Rightarrow \quad \frac{1}{k+2} + \frac{1}{k+3} + \dots\dots\dots + \frac{1}{2k+1} + \frac{1}{2k+2} > \frac{13}{24} + \frac{1}{2k+1} + \frac{1}{2k+2} - \frac{1}{k+1}$$

$$\Rightarrow \quad \frac{1}{k+2} + \dots\dots\dots + \frac{1}{2k+1} = \frac{1}{2k+2} > \frac{13}{24} + \frac{(2k+2)+(2k+1)-2(2k+1)}{2(k+1)(2k+1)}$$

$$\Rightarrow \quad \frac{1}{k+2} + \dots\dots\dots\dots + \frac{1}{2k+1} + \frac{1}{2k+2} > \frac{13}{24} + \frac{1}{2(k+1)(2k+1)}$$

as $\frac{1}{2(k+1)(2k+1)}$ is positive, it can be removed the smaller side

$$\Rightarrow \quad \frac{1}{k+2} + \dots\dots\dots + \frac{1}{2k+1} + \frac{1}{2k+2} > \frac{13}{24}$$

$\Rightarrow$ P(k + 1) is true

Hence P(2) is true and P(k + 1) is true whenever P(k) is true

$\Rightarrow$ By induction, P(n) is true for all $n > 1$, $n \in N$

Example : 42

If n > 1, prove that $n! < \left(\frac{n+1}{2}\right)^n$

Solution

Let P(n) : $n! < \left(\frac{n+1}{2}\right)^n$

for n = 2, $2! < \left(\frac{3}{2}\right)^2$ which is true

$\Rightarrow$ P(2) is true

Let P(k) be true

$\Rightarrow$ $k! < \left(\frac{k+1}{2}\right)^k$

P(k + 1) : $(k+1)! < \left(\frac{k+2}{2}\right)^{k+1}$ ....(i)

using P(k), we have

$k! < \left(\frac{k+1}{2}\right)^k$

$\Rightarrow$ $(k+1)! < \frac{(k+1)^{k+1}}{2^k}$ ....(ii)

Let us try to compare the RHS of (i) and (ii).

Let us assume that $\frac{(k+1)^{k+1}}{2^k} < \left(\frac{k+2}{2}\right)^{k+1}$ ....(iii)

$\Rightarrow$ $\left(\frac{k+2}{k+1}\right)^{k+1} > 2$ $\Rightarrow$ $\left(1+\frac{1}{k+1}\right)^{k+1} > 2$

Using Binomial Expansion :

$\Rightarrow$ $1 + (k+1)\frac{1}{k+1} + {}^{k+1}C_2\left(\frac{1}{k+1}\right)^2 + \ldots\ldots\ldots > 2$

$\Rightarrow$ $1 + 1 + {}^{k+1}C_2\left(\frac{1}{k+1}\right)^2 + \ldots\ldots\ldots > 2$ which is true

Hence (iii) is true

From (ii) and (iii), we have

$(k+1)! < \frac{(k+1)^{k+1}}{2^k} < \left(\frac{k+2}{2}\right)^{k+1}$

$\Rightarrow$ $(k+1)! < \left(\frac{k+2}{2}\right)^{k+1}$

P(K + 1) is true

Hence P(2) is true and P(k + 1) is true whenever P(k) is true

$\Rightarrow$ By induction, P(n) is true for all $n > 1$, $n \in N$

Example : 43

Prove that $A_n = \cos n\theta$ if it is given that $A_1 = \cos\theta$, $A_2 = \cos 2\theta$ and for every natural number m > 2, the relation $A_m = 2\,A_{m-1}\cos\theta - A_{m-2}$.

Solution

The principle of induction can be extended to the following form :

P(n) is true for all $n \in N$, if

(i) P(1) is true and P(2) is true and

(ii) P(k + 2) is true whenever P(k) and P(k + 1) are true

Let P(n) : $A_n = \cos n\theta$

Hence $A_1 = \cos\theta$, $A_2 = \cos 2\theta$ $\Rightarrow$ P(1) and P(2) are true

Now let us assume that P(k) and P(k + 1) are true

$\Rightarrow$ $A_k = \cos k\theta$ and $A_{k+1} = \cos(k+1)\,\theta$

We will now try to show that P(k + 2) is true

Using $A_m = 2\,A_{m-1}\cos\theta - A_{m-2}$, (for m > 2)

We have $A_{k+2} = 2A_{k+1}\cos\theta - A_k$ (for k > 0)

$$\Rightarrow \quad A_{k+2} = 2\cos(k+1)\,\theta\cos\theta = \cos k\theta$$

$$= \cos(k+2)\theta + \cos k\theta - \cos k\theta$$

$$= \cos(k+2)\,\theta$$

$\Rightarrow$ P(k + 2) is true

Hence P(1), P(2) are true and P(k + 2) is true whenever P(k), P(k + 1 are true

$\Rightarrow$ By induction, P(n) is true for all $n \in N$

Example : 44

Let $u_1 = 1$, $u_2 = 1$ and $u_{n+2} = u_n + u_{n+1}$ for $n \geq 1$. Use induction to show that $u_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right]$

for all $n \geq 1$.

Solution

$$\text{Let} \quad P(n) : u_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right]$$

$$P(1) : u_1 = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^1 - \left(\frac{1-\sqrt{5}}{2}\right)^1\right] = 1 \qquad \text{which is true}$$

$$P(2) : u_2 = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^2 - \left(\frac{1-\sqrt{5}}{2}\right)^2\right] = 1 \qquad \text{which is true}$$

Hence P(1), P(2) are true

Let P(k), P(k + 1) be true

$$\Rightarrow \quad \text{We have : } u_k = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^k - \left(\frac{1-\sqrt{5}}{2}\right)^k\right]$$

$$\text{And} \quad u_{k-1} = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{k+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{k+1}\right]$$

Let us try to prove that P(k + 2) is true

From the given relation : $u_{k+2} = u_k + u_{k+1}$

$$\Rightarrow \quad u_{k+2} = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^k - \left(\frac{1-\sqrt{5}}{2}\right)^k\right] - \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{k+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{k+1}\right]$$

$\Rightarrow \quad u_{k+2} = \dfrac{1}{\sqrt{5}}\left[\left(\dfrac{1+\sqrt{5}}{2}\right)^{k}\left(1+\dfrac{1+\sqrt{5}}{2}\right)\right] - \dfrac{1}{\sqrt{5}}\left[\left(\dfrac{1-\sqrt{5}}{2}\right)^{k}\left(1+\dfrac{1-\sqrt{5}}{2}\right)\right]$

$\Rightarrow \quad u_{k+2} = \dfrac{1}{\sqrt{5}}\left[\left(\dfrac{1+\sqrt{5}}{2}\right)^{k}\left(\dfrac{1+\sqrt{5}}{2}\right)^{2}\right] - \left[\left(\dfrac{1-\sqrt{5}}{2}\right)^{k}\left(\dfrac{1-\sqrt{5}}{2}\right)^{2}\right]$

$\Rightarrow \quad u_{k+2} = \dfrac{1}{\sqrt{5}}\left[\left(\dfrac{1-\sqrt{5}}{2}\right)^{k+2} - \left(\dfrac{1-\sqrt{5}}{2}\right)^{k+2}\right]$

$\Rightarrow$ P(k + 2) is true

Hence P(1), P(2) are true and P(k + 2) is true whenever P(k), P(k + 1) are true

$\Rightarrow$ By induction, P(n) is true for all $n \in N$

Example : 45

Use mathematical induction to prove that $\sum\limits_{k=0}^{n} k^2\ {}^nC_k = n(n + 1)\ 2^{n-2}$ for $n \geq 1$

Solution

Let P(n) : $\sum\limits_{k=0}^{n} k^2\ {}^nC_k = n\ (n + 1)\ 2^{n-2}$

for n = 1 : $\sum\limits_{k=0}^{n} k^2\ {}^1C_k = 1\ (1 + 1)\ 2^{1-2}$

i.e. 1 = 1 which is true $\Rightarrow$ P(1) is true

Let P(m) be true

$\Rightarrow \quad \sum\limits_{k=0}^{m} k^2\ {}^mC_k = m\ (m + 1)\ 2^{m-2}$

consider P(m + 1) : $\sum\limits_{k=0}^{m+1} k^2\ {}^{m+1}C_k = (m + 1)\ (m + 2)\ 2^{m-1}$

LHS of P(m + 1) : $= \sum\limits_{k=0}^{m+1} k^2\ {}^{m+1}C_k = \sum\limits_{k=0}^{m+1} k^2\ ({}^mC_k + {}^mC_{k-1}) = \sum\limits_{k=0}^{m} k^2\ {}^mC_k + \sum\limits_{k=1}^{m+1} k^2\ {}^mC_{k-1}$

$= m(m + 1)\ 2^{m-2} + \sum\limits_{t=0}^{m} (t+1)^2\ {}^mC_t$ substituting k = t + 1

$= m\ (m + 1)\ 2^{m-2} + \sum\limits_{t=0}^{m+1} t^2\ {}^mC_t + 2\sum\limits_{t=0}^{m} t\ {}^mC_t + \sum\limits_{t=0}^{m} {}^mC_t$

using P(k) and $C_1 + 2C_2 + 3C_3 + \ldots\ldots\ldots\ldots{}^nC_n = n2^{n-1}$

$\Rightarrow$ LHS = $m\ (m + 1)\ 2^{m-2} + m\ (m + 1)\ 2^{m-2} + 2\ (m2^{m-1}) + 2^m = 2^{m-1}\ [m(m + 1) + 2m + 2]$

$= 2^{m-1}\ (m + 1)\ (m + 2) =$ RHS

$\Rightarrow$ P(m + 1) is true

Hence P(1) is true and P(m + 1) is true whenever P(m) is true

$\Rightarrow$ By induction, P(n) is true for all $n \in N$

Example : 46

Using mathematical induction, prove $\frac{n^5}{5} + \frac{n^3}{3} + \frac{7n}{15}$ is an integer for all $n \in N$

Solution

Let P(n) : $\frac{n^5}{5} + \frac{n^3}{3} + \frac{7n}{15}$ is an integer

P(1) : $\frac{1}{5} + \frac{1}{3} = \frac{7}{15}$ = 1 is an integer $\Rightarrow$ P(1) is true

Let us assume that P(k) is true i.e. P(k) : $\frac{k^5}{5} + \frac{k^3}{3} + \frac{7k}{15}$ is an integer ....(i)

Consider LHS of P(k + 1)

$$\text{LHS of } P(k+1) = \frac{(k+1)^5}{5} + \frac{(k+1)^3}{3} + \frac{7(k+1)}{15}$$

$$= \frac{k^5+5k^4+10k^3+10k^2+5k+1}{5} + \frac{k^3+3k^2+3k+1}{3} + \frac{7(k+1)}{15}$$

$$= \frac{k^5}{5} + \frac{k^3}{3} + \frac{7k}{15} + k^4 + 2k^3 + 3k^2 + 2k + \frac{1}{5} = \frac{1}{3} + \frac{7}{15}$$

$= P(k) + k^4 + 2k^3 + 3k^2 + 2k + 1$ [using (i)]

As P(k) and k both are positive integers, we can conclude that P(k + 1) is also an integer

$\Rightarrow$ P(k + 1) is true

Hence by principle of mathematical induction, P(n) si true for all $n \in N$

Example : 47

Using mathematical induction, prove that for any non-negative integers n, m, r and k,

$$\sum_{m=0}^{k} (n-m)\,\frac{(r+m)!}{m!} = \frac{(r+k+1)!}{k!}\left[\frac{n}{r+1} - \frac{k}{r+2}\right]$$

Solution

In this problem, we will apply mathematical induction on k.

Let P(k) : $\sum_{m=0}^{k} (n-m)\,\frac{(r+m)!}{m!} = \frac{(r+k+1)!}{k!}\left[\frac{n}{r+1} - \frac{k}{r+2}\right]$

Consider P(0)

$$\text{LHS of } P(0) = \sum_{m=0}^{0} (n-m)\frac{(r+m)!}{m!} = n\,\frac{r!}{0!} = nr!$$

$$\text{RHS of } P(0) = \frac{(r+1)!}{0!}\left[\frac{n}{r+1} - \frac{0}{r+2}\right] = \frac{n(r+1)!}{r+1} = nr!$$

$\Rightarrow$ P(0) is true

Let us assume that P(k) is true for k = p

$$\Rightarrow \quad \sum_{m=0}^{p}(n-m)\,\frac{(r+m)!}{m!} = \frac{(r+p+1)!}{p!}\left[\frac{n}{r+1} - \frac{p}{r+2}\right] \quad ....(i)$$

Consider LHS of P(p + 1)

$$\text{LHS of } P(p+1) = \sum_{m=0}^{p+1} (n-m)\,\frac{(r+m!)}{m!} = \sum_{m=0}^{p} (n-m)\,\frac{(r+m)!}{m!} + (n-p-1)\,\frac{(r+p+1)!}{(p+1)!}$$

using (i), we get :

$$\text{LHS of } P(p+1) = \frac{(r+p+1)!}{p!}\left[\frac{n}{r+1}-\frac{p}{r+2}\right] + (n-p-1)\,\frac{(r+p+1)!}{(p+1)!}$$

$$= \frac{(r+p+1)!}{(p+1)!}\left[\frac{n(p+1)}{r+1}-\frac{(p+1)}{r+2}+n-(p+1)\right]$$

$$= \frac{(r+p+1)!}{(p+1)!}\left[\left(\frac{n(p+1)}{r+1}+n\right)-\left(\frac{(p+1)}{r+2}+(p+1)\right)\right]$$

$$= \frac{(r+p+1)!}{(p+1)!}\left[\frac{(p+r+2)n}{r+1}-\frac{(p+1)(p+r+2)}{r+2}\right]$$

$$= \frac{(r+p+2)!}{(p+1)!}\left[\frac{n}{r+1}-\frac{(p+1)}{r+2}\right] = \text{RHS of } P(p+1)$$

$\Rightarrow$ P(p + 1) is true

Hence, by principle of mathematical induction, P(n) is true for all n = 0, 1, 2, 3, ...

Example : 48

If x is not an integral multiple of 2π , use mathematical induction to prove that :

$$\cos x + \cos 2x + \dots\dots\dots + \cos nx = \cos\frac{n+1}{2}x\,\sin\frac{nx}{2}\,\text{cosec}\,\frac{x}{2}$$

Solution

Let P(n) : $\cos x + \cos 2x + \dots\dots\dots + \cos nx = \cos\frac{n+1}{2}x\,\sin\frac{nx}{2}\,\text{cosec}\,\frac{x}{2}$

LHS of P(1) = cos x

RHS of P(1) = $\cos\frac{1+1}{2}x\,\sin\frac{1.x}{2}\,\text{cosec}\,\frac{x}{2} = \cos x$

Let us assume that P(k) is true

i.e. P(k) : $\cos x + \cos 2x + \dots\dots\dots\dots + \cos kx = \cos\frac{k+1}{2}x\,\sin\frac{kx}{2}\,\text{cosec}\,\frac{x}{2}$

Consider LHS of P(k + 1)

LHS of P(k + 1) = cos x + cos 2x + ... + cos kx + cos (k + 1) x

Using P(k), we get :

$$\text{LHS of } P(k+1) = \cos\frac{k+1}{2}x\,\sin\frac{kx}{2}\,\text{cosec}\,\frac{x}{2} + \cos(k+1)x$$

$$= \frac{\cos\frac{k+1}{2}x\sin\frac{kx}{2}-\cos(k+1)x\sin\frac{x}{2}}{\sin\frac{x}{2}} = \frac{2\cos\frac{k+1}{2}x\sin\frac{kx}{2}-2\cos(k+1)x\sin\frac{x}{2}}{2\sin\frac{x}{2}}$$

$$= \frac{\sin\left(\frac{2k+1}{2}\right)x-\sin\frac{kx}{2}+\sin\left(\frac{2k+3}{2}\right)x-\sin\left(\frac{2k+1}{2}\right)x}{2\sin\frac{x}{2}} = \frac{\sin\left(\frac{2k+3}{2}\right)x-\sin\frac{kx}{2}}{2\sin\frac{x}{2}}$$

$$= \frac{2\cos\left(\frac{k+2}{2}\right)x\sin\left(\frac{k+1}{2}\right)x}{2\sin\frac{x}{2}} = \cos\left(\frac{k+2}{2}\right)x\,\sin\left(\frac{k+1}{2}\right)x\,\text{cosec}\,\frac{x}{2} = \text{RHS of } P(k+1)$$

$\Rightarrow$ P(k + 1) is true

Hence by principle of mathematical induction, P(n) is true for all $n \in N$

Example : 49

Using mathematical induction, prove that for every integer $n \geq 1$, $3^{2^n} - 1$ is divisible by 2^{n+2} but not divisible by 2^{n+3} .

Solution

Let P(n) : $3^{2^n} - 1$ is divisible by 2^{n+2} , but not divisible by 2^{n+3}.

P(1) : 8 is divisible by 2^3 , but not divisible by 2^4 .

$\Rightarrow$ P(1) : 8 is divisible by 8, but not divisible by 16

$\Rightarrow$ P(1) is true

Let P(k) is true

i.e. $3^{2^k} - 1$ is divisible by 2^{k+2} , but not divisible by 2^{k+3}

$\Rightarrow$ $3^{2^k} - 1 = m\,2^{k+2}$, where m is odd number so that P(k) is not divisible by 2^{k+3} ....(i)

Consider P(k + 1)

LHS of P (k + 1) = $3^{2^{k+1}} - 1 = \left(3^{2^k}\right)^2 - 1$

Using (i), we get :

LHS of P(k + 1) = $(m2^{k+2} + 1)^2 - 1$

$= m^2\, 2^{2k+4} + 2m.2^{k+2}$

$= 2^{k+3}\,(m^2\, 2^{k+1} + m)$

$= p\, 2^{k+3}$ where p is an odd number because $m^2\, 2^{k+1}$ is even and m is odd.

$\Rightarrow$ P(k + 1) is divisible by 2^{k+3} , but not divisible by 2^{k+4} as p is odd

$\Rightarrow$ P(k + 1) is true

Hence, by mathematical induction, P(n) is true for all $n \in N$

Example : 50

Using mathematical induction, prove that : ${}^mC_0\, {}^nC_k + {}^mC_1\, {}^nC_{k-1} + {}^mC_2\, {}^nC_{k-2} + \ldots\ldots + {}^mC_k\, {}^nC_0 = {}^{m+n}C_k$ for p < q, where m, n and k are possible integers and ${}^pC_q = 0$ for p < q.

Solution

First apply mathematical induction on n

Let P(n) : ${}^mC_0\, {}^nC_k + {}^mC_1\, {}^nC_{k-1} + {}^mC_2\, {}^nC_{k-2} + \ldots\ldots.. + {}^mC_k\, {}^nC_0 = {}^{m+n}C_k$

Consider P(1)

LHS of P(1) = ${}^mC_{k-1}\, {}^1C_1 + {}^mC_k\, {}^1C_0 = {}^{m+1}C_k$ = RHS of P(1)

$\Rightarrow$ P(1) is true

Assume that P(n) is true for n = s

i.e. P(s) : ${}^mC_0\, {}^sC_k + {}^mC_1\, {}^sC_{k-1} + {}^mC_2\, {}^sC_{k-2} + \ldots\ldots. + {}^mC_k\, {}^sC_0 = {}^{m+s}C_k$

Consider LHS of P(s + 1)

LHS of P(s + 1) = ${}^mC_0\, {}^{s+1}C_k + {}^mC_1\, {}^{s+1}C_{k-1} + {}^mC_2\, {}^{s+1}C_{k-2} + \ldots\ldots. + {}^mC_k\, {}^{s+1}C_0$

$\Rightarrow$ LHS of P(s + 1) = ${}^mC_0\, ({}^sC_k + {}^sC_{k-1}) + {}^mC_1\, ({}^sC_{k-1} + {}^sC_{k-2}) + \ldots\ldots.. + {}^mC_k\, {}^{s+1}C_0$

$= [{}^mC_0\, {}^sC_k + {}^mC_1\, {}^sC_{k-1} + \ldots\ldots. + {}^mC_k\, {}^sC_0] - [{}^mC_0\, {}^sC_{k-1} + {}^mC_1\, {}^sC_{k-2} + \ldots\ldots.. + {}^mC_{k-1}\, {}^sC_0]$

= P(s) + P(s)]where $_{\text{k is replaced by k – 1 in the P(s)}}$

$\Rightarrow$ LHS of P(s + 1) = ${}^{m+s}C_k + {}^{m+s}C_{k-1} = {}^{m+s+1}C_k$ = RHS of P(s + 1)

$\Rightarrow$ P(n + 1) is true for all $n \in N$

Similarly we can show that the given statement is true for all $m \in N$.

Example : 51

Let $p \geq 3$ be an integer and α, β be the roots of $x^2 - (p + 1)\,x + 1 = 0$. Using mathematical induction, show that $\alpha^n + \beta^n$

(i) is an integer and

(ii) is not divisible by p

Solution

It is given that α and β are roots of $x^2 - (p + 1)\,x + 1 = 0$

$\Rightarrow$ $\alpha + \beta = p + 1$ and $\alpha\beta = 1$ ....(i)

(i) Let P(n) : $\alpha^n + \beta^n$ is an integer

P(1) : $\alpha + \beta = p + 1$ is an integer
As it is given that p is an integer, P(1) is true.
P(2) : $\alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta = (p + 1)^2 - 2$ is an integer.
As p is an integer, $(p + 1)^2 - 2$ is also an integer ⇒ P(2) is true
Assume that both P(k) and P(k – 1) are true
i.e. $\alpha^k + \beta^k$ and $\alpha^{k-1} + \beta^{k-1}$ both are integers
Consider LHS of P(k + 1) i.e.
LHS of P(k + 1) = $\alpha^{k+1} + \beta^{k-1} = (\alpha - \beta)(\alpha^k + b^k) - \alpha\beta(\alpha^{k-1} + b^{k-1})$
⇒ LHS of P(k + 1) = p P(k) – P(k –1) [using (i)]
⇒ LHS of P(k + 1) = integer because p, P(k – 1) and P(k) all are integer
⇒ P(k + 1) is true. Hence P(n) is true for n ∈ N.

(ii) Let P(n) = $\alpha^n + \beta^n$ is not divisible by p
P(1) : $\alpha + \beta = p + 1$ = a number which is not divisible by p ⇒ P(1) is true
P(2) : $\alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta$
$= (p + 1)^2 - 2 = p(p + 2) - 1$
= a number which is divisible by p – a number which is not divisible by p
= a number which is not divisible by p ⇒ P(2) is true
P(3) : $\alpha^3 + b^3 = (\alpha + \beta)(\alpha^2 + \beta^2 - \alpha\beta) = (p + 1)[(p + 1)^2 - 3] = p[(p + 1)^2 - 3] + p(p + 2) - 2$
$= p[(p + 1)^2 + p - 1] - 2$
= a number which is divisible by p – a number which is not divisible by p
= a number which is not divisible p ⇒ P(3) is true
Assume that P(k), P(k – 1) and P(k – 2) all are true
i.e. $\alpha^k + \beta^k$, α^{k-1} and $\alpha^{k+2} + \beta^{k-2}$ all are non-divisible by p.
Consider LHS of P(k + 1) i.e.
LHS of P(k + 1) = $\alpha^{k+1} + \beta^{k+1} = (\alpha + \beta)(\alpha^k + b^k) - \alpha\beta(\alpha^{k-1} + b^{k-1})$
$= p(\alpha^k - b^k) + (\alpha^k + b^k) - (\alpha^{k-1} + b^{k-1})$
$= p\,P(k) + [(p + 1)(\alpha^{k-1} - b^{k-1}) - (\alpha^{k+2} + b^{k-2})] - (\alpha^{k-1} + b^{k-1})$
$= p\,P(k) + p\,P(k - 1) - P(k - 2)$
$= p[P(k) + P(k - 1)] - P(k - 2)$
= a number which is divisible by p – a number which is not divisible by p
= a number which is not divisible by p
⇒ P(k + 1) is true
Hence, by principle of mathematical induction P(n) is true for all n ∈ N

Example : 52

Use mathematical induction to prove that $\frac{d^n}{dx^n}\left(\frac{\log x}{x}\right) = \frac{(-1)^n}{x^{n+1}}\left(\log x - 1 - \frac{1}{2} - ...... - \frac{1}{n}\right)$ for all n ∈ N and x > 0.

Solution

Let P(n) : $\frac{d^n}{dx^n}\left(\frac{\log x}{x}\right) = \frac{(-1)^n}{x^{n+1}}\left(\log x - 1 - \frac{1}{2} - ...... - \frac{1}{n}\right)$

LHS of P(1) : $\frac{d}{dx}\left(\frac{\log x}{x}\right) = \frac{\frac{1}{x} . x - \log x}{x^2} = \frac{1 - \log x}{x^2}$

RHS of P(1) = $\frac{(-1)!}{x^2}$ (log x – 1) = $\frac{1 - \log x}{x^2}$

⇒ P(1) is true
Let us assume that P(k) is true i.e.

P(k) : $\frac{d^k}{dx^k}\left(\frac{\log x}{x}\right) = \frac{(-1)^k k!}{x^{k-1}}\left(\log x - 1 - \frac{1}{2} - ...... - \frac{1}{k}\right)$ ....(i)

Consider LHS of P(k + 1) i.e.

LHS of P(k + 1) = $\frac{d^{k+1}}{dx^{k+1}}\left(\frac{\log x}{x}\right) = \frac{d}{dx}\left[\frac{d^k}{dx^k}\left(\frac{\log x}{x}\right)\right]$

$= \frac{d}{dx}$ [LHS of P(k)] = $\frac{d}{dx}$ [RHS of P(k)]   [using (1)]

$$= \frac{d}{dx}\left[\frac{(-1)^k k!}{x^{k+1}}\left(\log x - 1 - \frac{1}{2} - \ldots\ldots\ldots - \frac{1}{k}\right)\right]$$

$$= \frac{(-1)^k k!(-1)(k+1)}{x^{k+2}}\left(\log x - 1 - \frac{1}{2} - \ldots\ldots\ldots - \frac{1}{k}\right) + \frac{(-1)^k k!}{x^{k+1}}\,\frac{1}{x}$$

$$= \frac{(-1)^{K+1}(K+1)!}{X^{K+2}}\left(\log x - 1 - \frac{1}{2} - \ldots\ldots - \frac{1}{k+1}\right)$$

⇒ P(k + 1) is true

Hence by principle of mathematical induction, P(n) is true for all $n \in N$

Example : 53

Use mathematical induction to prove that $\frac{d^n}{dx^n}$ $(x^n \log x) = n!\left(\log x + 1 + \frac{1}{2} + \ldots.. + \frac{1}{n}\right)$ for all $n \in N$ and $x > 0$.

Solution

Let P(n) : $\frac{d^n}{dx^n}(x^n \log x) = n!\left(\log x + 1 + \frac{1}{2} + \ldots.. + \frac{1}{n}\right)$

LHS of P(1) = $\left(\frac{d}{dx}\right)(x \log x) = \log x + \frac{x}{x} = \log x + 1$

RHS of P(1) = 1! (log x + 1) = log x + 1

⇒ P(1) is true

Let us assume that P(k) is true i.e.

P(k) : $\frac{d^k}{dx^k}(x^k \log x) = k!\left(\log x + 1 + \frac{1}{2} + \ldots\ldots.. + \frac{1}{k}\right)$ ....(i)

Consider LHS of P(k + 1)   i.e.

LHS of P(k + 1) = $\frac{d^{k+1}}{dx^{k+1}}(x^{k+1} \log x)$

$$= \frac{d^k}{dx^k}\left[\frac{d}{dx}(x^{k+1}\log x)\right]$$

$$= \frac{d^k}{dx^k}\left[(k+1)x^k \log x + \frac{x^{k+1}}{x}\right]$$

$$= (k + 1)\,\frac{d^k}{dx^k}[x^k \log x] + \frac{d^k}{dx^k}\left[\frac{x^{k+1}}{x}\right]$$

$$= (k + 1)\left[k!(\log x + 1 + \frac{1}{2} + \ldots.. + \frac{1}{k}\right] + k! \quad \text{[using (i)]}$$

$$= (k + 1)!\left[k!(\log x + 1 + \frac{1}{2} + \ldots\ldots + \frac{1}{k+1}\right]$$

$\Rightarrow$ P(k + 1) is true

Hence by principle of mathematical induction, P(n) is true for all $n \in N$

Example : 54

Use mathematical induction to prove $\int_0^{\pi/2} \frac{\sin^2 nx}{\sin x}\, dx = 1 + \frac{1}{3} + \frac{1}{5} + \dots\dots\dots + \frac{1}{2n-1}$ for all $n \in N$.

Solution

Consider $I_n = \int_0^{\pi/2} \frac{\sin^2 nx}{\sin x}\, dx = 1 + \frac{1}{3} + \frac{1}{5} + \dots\dots\dots + \frac{1}{2n-1}$

from left hand side, $I_1 = \int_0^{\pi/2} \frac{\sin^2 nx}{\sin x}\, dx = \int_0^{\pi/2} \sin x\, dx = 1$

from right hand side, $I_1 = 1$

$\Rightarrow$ I_1 is true

Assume that I_k is true i.e.

$$I_k = \int_0^{\pi/2} \frac{\sin^2 kx}{\sin x}\, dx = 1 + \frac{1}{3} + \frac{1}{5} + \dots\dots\dots + \frac{1}{2k-1} \qquad \dots\dots\dots\text{(i)}$$

Consider $I_{k-1} - I_k = \int_0^{\pi/2} \frac{\sin^2 (k+1)x}{\sin x}\, dx - \int_0^{\pi/2} \frac{\sin^2 kx}{\sin x}\, dx$

$$\Rightarrow \quad I_{k+1} - I_k = \int_0^{\pi/2} \frac{\sin^2 (k+1)x - \sin^2 kx}{\sin x}\, dx = \int_0^{\pi/2} \frac{\sin(2k+1)x \sin x}{\sin x}\, dx$$

$$= \int_0^{\pi/2} \sin(2k+1)x\, dx = -\frac{\cos(2k+1)x}{2k+1}\Bigg]_0^{\pi/2} = \frac{1}{2k+1}$$

$\Rightarrow$ $I_{k+1} = I_k + \frac{1}{2k+1}$ $\Rightarrow$ $I_{k+1} = I_k + \frac{1}{2k+1}$

$\Rightarrow$ $I_{k+1} = 1 + \frac{1}{3} + \frac{1}{5} + \dots\dots\dots + \frac{1}{2k-1} + \frac{1}{2k+1}$ [using (i)]

$\Rightarrow$ I_{k+1} is true.

Hence by principle of mathematical induction I_n is true for all values of $n \in N$

Example : 55

Let $I_n = \int_0^{\pi} \frac{1-\cos nx}{1-\cos x}\, dx$. Use mathematical induction to prove that $I_n = n\pi$ for all n = 0, 1, 2, 3, ...

Solution

We have to prove $I_n = \int_0^{\pi} \frac{1-\cos nx}{1-\cos x}\, dx = n\pi$

For n = 0

$$I_0 = \int_0^{\pi} \frac{1-\cos\theta}{1-\cos x}\, dx = \int_0^{\pi} 0\, dx = 0.$$

The value of the integral from the RHS = 0 × π = 0

⇒ The given integral is true for n = 0

From n = 1

$$I_1 = \int_0^\pi \frac{1-\cos x}{1-\cos x}\, dx = \int_0^\pi dx = \pi$$

The value of the integral from the RHS = 1 × π = π

⇒ The given integral is true for n = 1

Assume that the given integral is true for n = k – 1 and n = k i.e.

$$I_{k-1} = \int_0^\pi \frac{1-\cos(k-1)x}{1-\cos x}\, dx = (k-1)\,\pi \qquad \text{.........(i)}$$

$$I_k = \int_0^\pi \frac{1-\cos kx}{1-\cos x}\, dx = k\pi \qquad \text{.........(ii)}$$

Consider $I_{k+1} - I_k = \displaystyle\int_0^\pi \frac{\cos kx - \cos(k+1)x}{1-\cos x}\, dx$

$$\Rightarrow \quad I_{k+1} - I_k = \int_0^\pi \frac{2\sin\frac{x}{2}\sin\frac{2k+1}{2}x}{2\sin^2\frac{x}{2}}\, dx = \int_0^\pi \frac{\sin\frac{2k+1}{2}x}{\sin\frac{x}{2}}\, dx \qquad \text{..........(iii)}$$

Consider $I_k - I_{k-1} = \displaystyle\int_0^\pi \frac{\cos(k-1)x - \cos kx}{1-\cos x}\, dx$

$$\Rightarrow \quad I_k - I_{k-1} = \int_0^\pi \frac{2\sin\frac{x}{2}\sin\frac{2k-1}{2}x}{2\sin^2\frac{x}{2}}\, dx = \int_0^\pi \frac{\sin\frac{2k-1}{2}x}{\sin\frac{x}{2}}\, dx \qquad \text{..........(iv)}$$

Subtracting (iv) from (iii), we get :

$$I_{k+1} - 2\,I_k + I_{k-1} = \int_0^\pi \frac{\sin\frac{2k+1}{2}x - \sin\frac{2k-1}{2}x}{\sin\frac{x}{2}}\, dx$$

$$\Rightarrow \quad I_{k-1} - 2\,I_k + I_{k-1} = \int_0^\pi \frac{2\cos kx \sin\frac{x}{2}}{\sin\frac{x}{2}}\, dx = 2\int_0^x \cos kx\, dx = 2\left.\frac{\sin kx}{k}\right]_0^\pi = 0$$

⇒ $I_{k+1} = 2I_k - I_{k-1} = 2k\pi - (k-1)\pi$ [using (i) and (ii)]

⇒ $I_{k+1} = (k+1)\,\pi$

⇒ The given integral is true for n = k + 1

Hence, by principle of mathematical induction, the given integral is true for all n = 0, 1, 2, 3, ...