Example : 1

Find the value of t so that the points (1, 1), (2, –1), (3, –2) and (12, t) are concyclic.

Solution

Let A ≡ (1, 1) B ≡ (2, –1) C ≡ (3, –2) D ≡ (12, t)

We will find the equation of the circle passing through A, B and C and then find t so that D lies on that circle. Any circle passing through A, B can be taken as :

$$(x-1)(x-2)+(y-1)(y+1)+k\begin{vmatrix} x & y & 1 \\ 1 & 1 & 1 \\ 2 & -1 & 1 \end{vmatrix}=0$$

⇒ $x^2 + y^2 - 3x + 1 + (2x + y - 3) = 0$

C ≡ (3, –2) lies on this circle.

⇒ $9 + 4 - 9 + 1 + k(6 - 2 - 3) = 0$

⇒ $k = -5$

⇒ circle through A, B and C is :

$x^2 + y^2 - 3x + 1 - 5(2x + y - 3) = 0$

$x^2 + y^2 - 12x - 5y + 16 = 0$

D ≡ (12, t) will lie on this circle if :

⇒ $144 + t^2 - 156 - 5t + 16 = 0$

⇒ $t^2 - 5t + 4 = 0$

⇒ y = 1, 4

⇒ for t = 1, 4 the points are concyclic

Example : 2

Find the equation of a circle touching the line x + 2y = 1 at the point (3, –1) and passing through the point (2, 1).

Solution

The equation of any circle touching x + 2y – 1 = 0 at the point (3, –1) can be taken as :

$(x-3)^2 + (y-1)^2 + k(x - 2y - 1) = 0$ (using result 5 from family of circles)

As the circle passes through (2, –1) :

$(2-3)^2 + (1+1)^2 + k(2 + 2 - 1) = 0$

⇒ $k = -5/3$

⇒ the required circle is : $3(x^2 + y^2) - 23x - 4y + 35 = 0$

Notes :

1. Let A ≡ (3, –1) and B ≡ (2, 1)

Let L_1 be the line through A perpendicular to x + 2y = 1. Let L_2 be the right bisector of AB. The centre of circle is the point of intersection of L_1 and L_2 . The equation of the circle can be found by this method also.

2. Let (h, k) be the centre of the circle.

The centre (h, k) can be found from these equations.

$$\Rightarrow \quad \frac{|h+2k-1|}{\sqrt{5}} = \sqrt{(h-3)^2+(k+1)^2} = \sqrt{(h-2)^2+(k-1)^2}$$

The centre (h, k) can be found from these equations

Example : 3

Find the equation of a circle which touches the Y-axis at (0, 4) and cuts an intercept of length 6 units on X-axis.

Solution

The equation of circle touching x = 0 at (0, 4) can be taken as :

$(x-0)^2 + (y-4)^2 + k(x) = 0$

$x^2 + y^2 + kx - 8y + 16 = 0$

The circle cuts X-axis at points $(x_1, 0)$ and $(x_2, 0)$ given by :

$x^2 + kx + 16 = 0$

X-intercept = difference of roots of this quadratic :

$6 = |x_2 - x_1|$

⇒ $36 = (x_2 + x_1)^2 - 4x_2 x_1$

$\Rightarrow$ $36 = k_2 - 4\ (16)$

$\Rightarrow$ $k = \pm 10$

Hence the required circle is : $x^2 + y^2 \pm 10x - 8y + 16 = 0$

Note :

1. If a circle of radius r touches the X-axis at (1, 0), the centre of the circle is (a, ± r)

2. If a circle of radius r touches the Y-axis at (0, b), the centre of the circle is (± r, b).

Example : 4

Find the equation of the circle passing through the points (4, 3) and (3, 2) and touching the line $3x - y - 17 = 0$

Solution

Using result 4 from the family of circles, any circle passing through $A \equiv (4, 3)$ and $B \equiv (3, 2)$ can be taken as :

$$(x - 4)(x - 3) + (y - 3)(y - 2) + k \begin{vmatrix} x & y & 1 \\ 4 & 3 & 1 \\ 3 & 2 & 1 \end{vmatrix} = 0$$

$x^2 + y^2 - 7x - 5y + 18 + k\,(x - y - 1) = 0$

This circle touches $3x - y - 17 = 0$

centre $\equiv \left(\frac{7-k}{2}, \frac{k+5}{2}\right)$ and radius $= \sqrt{\frac{(7-k)^2}{4} + \frac{(k+5)^2}{4} - (18-k)}$

For tangency, distance of centre from line $3x - y - 17 = 0$ is radius

$$\Rightarrow \quad \frac{\left|3\left(\frac{7-k}{2}\right) - \left(\frac{k+5}{2}\right) - 17\right|}{\sqrt{9+1}} = \sqrt{\frac{(7-k)^2}{4} + \frac{(k+5)^2}{4} - 18 + k}$$

$$\Rightarrow \quad \left(\frac{-4k-18}{\sqrt{10}}\right)^2 = (7 - k)^2 + (k + 5)^2 - 72 + 4k$$

$\Rightarrow$ $4(4k^2 + 81 + 36k) = 10\,(2k^2 + 2)$

$\Rightarrow$ $k^2 - 36k - 76 = 0$ $\quad\Rightarrow\quad$ $k = -2, 38$

$\Rightarrow$ there are two circles through A and B and touching $3x - y - 17 = 0$. The equation are :

$x^2 + y^2 - 7x - 5y + 18 - 2\,(x - y - 1) = 0$ and

$x^2 + y^2 - 7x - 5y + 18 + 38\,(x - y - 1) = 0$

$\Rightarrow$ $x^2 + y^2 - 9x - 3y + 20 = 0$ and

$x^2 + y^2 + 31x - 43y - 20 = 0$

Notes :

1. Let $C \equiv (h, k)$ be the centre of required circle and $M \equiv (7/2, 5/2)$ be the mid point of AB.

C lies on right bisector of AB

$\Rightarrow$ slope (CM) = slope (AB) = – 1

$$\Rightarrow \quad \left(\frac{k - 5/2}{h - 7/2}\right) \times (1) = -1$$

Also CA = distance of centre from $(3x - y - 17 = 0)$

$$\Rightarrow \quad \sqrt{(h-4)^2 (k-3)^2} = \frac{|3h - k - 17|}{\sqrt{10}}$$

We can get h, k from these two equations.

Example : 5

Find the points on the circle $x^2 + y^2 = 4$ whose distance from the line $4x + 3y = 12$ is 4/5 units

Solution

Let A, B be the points on $x^2 + y^2 = 4$ lying at a distance 4/5 from $4x + 3y = 12$

$\Rightarrow$ AB will be parallel to $4x + 3y = 12$

Let the equation of AB be : $4x + 3y = c$

distance between the two lines is : $\frac{|c-12|}{\sqrt{9+16}} = \frac{4}{5}$

$\Rightarrow$ c = 16, 8

$\Rightarrow$ the equation of AB is : $5x + 3y = 8$ and $4x + 3y = 16$

The points A, B can be found by solving for points of intersection of x2 + y2 = 4 with AB.

AB $\equiv$ $(4x + 3y - 8 = 0)$

$\Rightarrow$ $x^2 + \left(\frac{8-4x}{3}\right)^2 = 4$

$\Rightarrow$ $25x^2 - 64x + 28 = 0$

$\Rightarrow$ x = 2, 14/25

$\Rightarrow$ y = 0, 448/25

AB $\equiv$ $(4x + 3y - 16 = 0)$

$\Rightarrow$ $x^2 + \left(\frac{16-4x}{3}\right)^2 = 4$

$\Rightarrow$ $25x^2 - 128x + 220 = 0$

$\Rightarrow$ $D < 0$ $\Rightarrow$ no real roots

Hence there are two points on circle at distance 4/5 from line.

A $\equiv$ (2, 0) and B $\equiv$ (14/25, 48/25)

Alternate Method :

Let P $\equiv$ (2 cos,2 sin) be the point on the circle $x^2 + y^2 = 4$ distant 4/5 from given line.

The distance from line = 4/5.

$\Rightarrow$ $\frac{|4(2\cos\theta)+3(2\sin\theta)-12|}{5} = \frac{4}{5}$

Solve for θ to get the point P.

Example : 6

Find the equation of circle passing through (–2, 3) and touching both the axes.

Solution

As the circle touches both the axes and lies in the IInd quadrant, its centre is :

C $\equiv$ (–r, r), where r is the radius

Distance of centre from (–2, 3) = radius

$\Rightarrow$ $\sqrt{(r-2)^2+(3-r)^2} = r$

$\Rightarrow$ $r = 5 \pm 2\sqrt{3}$

$\Rightarrow$ the circles are : $(x + r)^2 + (y - r)^2 = r^2$

$\Rightarrow$ $x^2 + y^2 + 2(5 \pm 2\sqrt{3})\,x - 2\,(5 \pm 2\sqrt{3})\,y + (5 \pm 2\sqrt{3})^2 = 0$

Example : 7

Tangents PA and PB are drawn from the point P(h, k) to the circle $x^2 + y^2 = a^2$. Find the equation of circumcircle of ΔPAB and the area of ΔPAB

Solution

AB is the chord of contact for point P.

Equation of AB is : $hx + ky = a^2$

The circumcircle of ΔPAB passes through the intersection of circle

$x^2 + y^2 - a^2 = 0$ and the line $hx + ky - a^2 = 0$

Using S + k L = 0, we can write the equation of the circle as :
$(x^2 + y^2 - a^2) + k(hx + ky - a^2) = 0$ where k is parameter
As this circle passes through P(h, k) ;
$\Rightarrow \quad h^2 + k^2 - a^2 + k(h^2 + k^2 - a^2) = 0$
$\Rightarrow \quad k = -1$
The circle is $x^2 + y^2 - hx - ky = 0$
Area of $\Delta PAB = 1/2\,(PM) \times (AB)$ (PM is perpendicular to AB)

PM = distance of P from AB = $\dfrac{|h^2 + k^2 - a^2|}{\sqrt{h^2 + k^2}}$

PA = length of tangent from P = $\sqrt{h^2 + k^2 - a^2}$

Area = $\dfrac{1}{2}$ PM $\left[2\sqrt{PA^2 - PM^2}\right]$ = PM $\sqrt{PA^2 - PM^2}$

Area = $\dfrac{|h^2 + k^2 - a^2|}{\sqrt{h^2 + k^2}} \; \dfrac{a\left(\sqrt{h^2 + k^2 - a^2}\right)}{\sqrt{h^2 + k^2}}$

Area = $\dfrac{a(h^2 + k^2 - a^2)^{3/2}}{h^2 + k^2}$

Note that $h^2 + k^2 - a^2 > 0 \quad \because \quad$ (h, k) lies outside the circle

Example : 8

Examine if the two circles $x^2 + y^2 - 8y - 4 = 0$ and $x^2 + y^2 - 2x - 4y = 0$ touch each other. Find the point of contact if they touch.

Solution

For $\quad x^2 + y^2 - 2x - 4y = 0 \quad$ centre $C_1 \equiv (1, 2)$
and $\quad x^2 + y^2 - 8y - 4 = 0 \quad$ centre $C_2 \equiv (0, 4)$

using $r = \sqrt{g^2 + f^2 - c}$: $r_1 = \sqrt{5}$ and $r_2 = 2\sqrt{5}$

Now $\quad C_1 C_2 = \sqrt{(0-1)^2 + (4-2)^2} = \sqrt{5}$

$\Rightarrow \quad r_2 - r_1 = 2\sqrt{5} - \sqrt{5} = \sqrt{5}$
$\Rightarrow \quad C_1 C_2 = r_2 - r_1$
$\Rightarrow$ the circle touch internally
For point of contact :

Let P(x, y) be the point of contact. P divides $C_1 C_2$ externally in the ratio of $\sqrt{5} : 2\sqrt{5} \equiv 1 : 2$
using section formula, we get :

$x = \dfrac{1(0) - 2(1)}{1 - 2} = 2$

$y = \dfrac{1(4) - 2(2)}{1 - 2} = 0$

$\Rightarrow \quad P(x, y) \equiv (2, 0)$ is the point of contact

Example : 9

Find the equation of two tangents drawn to the circle $x^2 + y^2 - 2x + 4y = 0$ from the point (0, 1)

Solution

Let m be the slope of the tangent. For two tangents there will be two values of m which are required
As the tangent passes though (0, 1), its equation will be :
$y - 1 = m(x - 0) \quad \Rightarrow \quad mx - y + 1 = 0$

Now the centre of circle $(x^2 + y^2 - 2x + 4y = 0) \equiv (1, -2)$ and $r = \sqrt{5}$
So using the condition of tangency : distance of centre (1, –2) from line = radius (r)

$$\frac{|m(1)-(-2)+1|}{\sqrt{m^2+1}} = \sqrt{5}$$

⇒ $(3 + m)^2 = 5(1 + m^2)$ ⇒ $m = 2, -1/2$

⇒ equations of tangents are :

$2x - y + 1 = 0$ (slope = 2) and $x + 2y - 2 = 0$ (slope = –1/2)

Example : 10

Find the equations of circles with radius 15 and touching the circle $x^2 + y^2 = 100$ at the point (6, –8).

Solution

Case – 1 :

If the require circle touches $x^2 + y^2 = 100$ at (6, –8) externally, then P(6, –8) divides OA in the ratio 2 : 3 internally.

Let centre of the circle be (h, k). Now using section formula :

⇒ $\frac{2k+3(0)}{2+3} = 6$

⇒ $\frac{2k+3(0)}{2+3} = -8$

⇒ $k = 15$ and $k = -20$

⇒ $(x - 15)^2 + (y + 20)^2 = 225$ is the required circle.

Case – 2 :

If the required circle touches $x^2 + y^2 = 100$ at (6, –8) internally, then P(6, –8) divides OA in the ratio 2 : 3 externally. Let centre of the circle be (h, k). Now using section formula :

⇒ $\frac{2h-3(0)}{2-3} = 6$

⇒ $\frac{2k-3(0)}{2-3} = -8$

⇒ $h = -3$ and $k = 4$

⇒ $(x + 3)^2 + (y - 4)^2 = 225$ is the required circle.

Example : 11

For what values of m, will the line y = mx does not intersect the circle $x^2 + y^2 + 20x + 20y + 20 = 0$?

Solution

If the line y = mx does not intersect the circle, the perpendicular distance of the line from the centre of the circle must be greater than its radius.

Centre of circle ≡ (–10, –10) ; radius $r = 6\sqrt{5}$

distance of line mx – y = 0 from (–10, –10) = $\frac{|m(-10)-(-10)|}{\sqrt{m^2+1}}$

⇒ $\frac{|10-10m|}{\sqrt{m^2+1}} > 6\sqrt{5}$

⇒ $(2m + 1)(m + 2) < 0$

⇒ $-2 < m < -1/2$

Example : 12

Find the equation of circle passing through (–4, 3) and touching the lines x + y = 2 and x – y = 2.

Solution

Let (h, k) be the centre of the circle. The distance of the centre from the given line and the given point must be equal to radius

⇒ $\frac{|h+k-2|}{\sqrt{2}} = \frac{|h-k-2|}{\sqrt{2}} = \sqrt{(h+4)^2+(k-3)^2}$

Consider $\frac{|h+k-2|}{\sqrt{2}} = \frac{|h-k-2|}{\sqrt{2}}$

$\Rightarrow$ $h + k - 2 = \pm (h - k - 2)$

Case 1 : (k = 0)

$\frac{|h-2|}{\sqrt{2}} = \sqrt{(h+4)^2+9}$

$(h-2)^2 = 2(h+4)^2 + 18$ $\Rightarrow$ $h^2 + 20h + 46 = 0$

$\Rightarrow$ $h = -10 \pm 3\sqrt{6}$

radius = $\frac{|h+k-2|}{\sqrt{2}} = \left|\frac{-12\pm3\sqrt{6}}{\sqrt{2}}\right|$

$\Rightarrow$ circle is : $(x + 10 \mp 3\sqrt{6})^2 + (y-0)^2 = \frac{(-12\pm3\sqrt{6})^2}{2}$

$\Rightarrow$ $x^2 + y^2 + 2(10 \pm 3\sqrt{6})x + (10 \pm 3\sqrt{6})^2 - \frac{(-12\pm3\sqrt{6})^2}{2} = 0$

$\Rightarrow$ $x^2 + y^2 + 2(10 \pm 3\sqrt{6})x + 55 \pm 24\sqrt{6} = 0$

Case – 2 : (h = 2)

$\frac{|k|}{\sqrt{2}} = \sqrt{36+(k-3)^2}$

$\Rightarrow$ $k^2 = 72 + 2(k-3)^2$ $\Rightarrow$ $k^2 - 12k + 90 = 0$

The equation has no real roots. Hence no circle is possible for h = 2

Hence only two circles are possible (k = 0)

$x^2 + y^2 + 2(10 \pm 3\sqrt{6})x + 55 \pm 24\sqrt{6} = 0$

Example : 13

The centre of circle S lies on the line $2x - 2y + 9 = 0$ and S cuts at right angles the circle $x^2 + y^2 = 4$. Show that S passes through two fixed points and find their coordinates.

Solution

Let the circle S be : $x^2 + y^2 + 2gx + 2fy + c = 0$

centre lies on $2x - 2y + 9 = 0$

$\Rightarrow$ $-2g + 2f + 9 = 0$ ....(i)

S cuts $x^2 + y^2 - 4 = 0$ orthogonally,

$\Rightarrow$ $2g(0) + 2f(0) = c - 4$

$\Rightarrow$ $c = 4$ ....(ii)

Using (i) and (ii) the equation of S becomes :

$x^2 + y^2 + (2f + 9)x + 2fy + 4 = 0$

$\Rightarrow$ $(x^2 + y^2 + 9x + 4) + f(2x + 2y) = 0$

We can compare this equation with the equation of the family of circle though the point of intersection of a circle and a line (S + fL = 0, where f is a parameter).

Hence the circle S always passes through two fixed points A and B which are the points of intersection of $x^2 + y^2 + 9x + 4 = 0$ and $2x + 2y = 0$

Solving these equations, we get :

$x^2 + x^2 + 9x + 4 = 0$

$\Rightarrow$ $x = -4, -1/2$ $\Rightarrow$ $y = 4, 1/2$

$\Rightarrow$ $A \equiv (-4, 4)$ and $B \equiv (-1/2, 1/2)$

Example : 14

A tangent is drawn to each of the circle $x^2 + y^2 = a^2$, $x^2 + y^2 = b^2$. Show that if the two tangents are perpendicular to each other, the locus of their point of intersection is a circle concentric with the given circles.

Solution

Let $P \equiv (x_1, y_1)$ be the point of intersection of the tangents PA and PB where A, B are points of contact with the two circles respectively.

As PA perpendicular to PB, the corresponding radii OA and OB are also perpendicular.

Let $\angle AOX = \theta$

$\Rightarrow \quad \angle BOX = \theta + 90^o$

Using the parametric form of the circles we can take :

$A \equiv (a \cos \theta , a \sin \theta)$

$B \equiv [b \cos (\theta + 90^o), b \sin (\theta + 90^o)]$

$B \equiv (-b \sin \theta, b \cos \theta)$

The equation of PA is : $x (a \cos \theta) + y (a \sin \theta) = a^2$

$\Rightarrow \quad x \cos \theta + y \sin \theta = a$

The equation of PB is :

$x(-b \sin \theta) + y (b \cos \theta) = b^2$

$\Rightarrow \quad y \cos \theta - x \sin \theta = b$

$\Rightarrow \quad P \equiv (x_1, y_1)$ lies on PA and PB both

$\Rightarrow \quad x_1 \cos \theta + y_1 \sin \theta = a$ and $y_1 \cos \theta - x_1 \sin \theta = b$

As θ is changing quantity (different for different positions of P), we will eliminate.

Squaring and adding, we get :

$x_1^2 + y_1^2 = a^2 + b^2$

$\Rightarrow$ the locus of P is $x^2 + y^2 = a^2 + b^2$ which is concentric with the given circles.

Example : 15

Secants are drawn from origin to the circle $(x - h)^2 + (y - k)^2 = r^2$. Find the locus of the mid-point of the portion of the secants intercepted inside the circle.

Solution

Let $C \equiv (h, k)$ be the centre of the given circle and $P \equiv (x_1 , y_1)$ be the mid-point of the portion AB of the secant OAB.

$\Rightarrow \quad CP \perp AB$

$\Rightarrow$ slope (OP) × slope (CP) = – 1

$$\Rightarrow \quad \left(\frac{y_1 - 0}{x_1 - 0}\right) \times \left(\frac{y_1 - k}{x_1 - h}\right) = -1$$

$\Rightarrow \quad x_1^2 + y_1^2 - hx_1 - ky_1 = 0$

$\Rightarrow$ the locus of the point P is : $x^2 + y^2 - hx - ky = 0$

Example : 16

The circle $x^2 + y^2 - 4x - 4y + 4 = 0$ is inscribed in a triangle which has two of its sides along the coordinate axes. The locus of the circumcentre of the triangle is $x + y - xy + k (x^2 + y^2)^{1/2} = 0$. Find value of k.

Solution

The given circle is $(x - 2)^2 + (y - 2)^2 = 4$

$\Rightarrow$ centre = (2, 2) and radius = 2

Let OAB be the triangle in which the circle is inscribed. As ΔOAB is right angled, the circumcentre is mid-point of AB.

Let $P \equiv (x_1, y_1)$ be the circumcentre.

$\Rightarrow \quad A \equiv (2x_1, 0)$ and $B \equiv (0, 2y_1)$

$$\Rightarrow \quad \text{the equation of AB is : } \frac{x}{2x_1} + \frac{y}{2y_1} = 1$$

As ΔAOB touches the circle, distance of C from AB = radius

$\Rightarrow \qquad \dfrac{\left|\dfrac{2}{2x_1}+\dfrac{2}{2y_1}-1\right|}{\sqrt{\dfrac{1}{4x_1^2}+\dfrac{1}{4y_1^2}}} = 2 \qquad$ ....(i)

As the centre (2, 2) lies on the origin side of the line $\dfrac{x}{2x_1} + \dfrac{y}{2y_1} - 1 = 0$

the expression $\dfrac{2}{2x_1} + \dfrac{2}{2y_1} - 1$ has the same sign as the constant term (–1) in the equation

$\Rightarrow \qquad \dfrac{2}{2x_1} + \dfrac{2}{2y_1} - 1$ is negative

$\Rightarrow$ equation (i) is : $\quad -\left(\dfrac{2}{2x_1}+\dfrac{2}{2y_1}-1\right) = 2\sqrt{\dfrac{1}{4x_1^2}+\dfrac{1}{4y_1^2}}$

$\Rightarrow \qquad -(x_1 + y_1 - x_1y_1) = \sqrt{x_1^2+y_1^2}$

$\Rightarrow$ the locus is : $x + y - xy + \sqrt{x^2+y^2} = 0$

$\Rightarrow \qquad k = 1$

Alternate Solution

We know, $\qquad r = \Delta/S$ where r is inradius, Δ is the area triangle and S is the semi-perimeter

$\Rightarrow \qquad 2 = \dfrac{\frac{1}{2}(2x_1)(2y_1)}{\dfrac{2x_1+2y_1+\sqrt{4x_1^2+4y_1^2}}{2}}$

$\Rightarrow \qquad 2 = \dfrac{\frac{1}{2}(2x_1)(2y_1)}{\dfrac{2x_1+2y_1+\sqrt{4x_1^2+4y_1^2}}{2}}$

$\Rightarrow \qquad 2 = \dfrac{2x_1y_1}{x_1+y_1+\sqrt{x_1^2+y_1^2}}$

$\Rightarrow \qquad x_1 + y_1 - x_1y_1 + \sqrt{x_1^2+y_1^2} = 0$

$\Rightarrow$ the locus is : $x + y - xy = \sqrt{x^2+y^2} = 0 \quad \Rightarrow \quad k = 1$

Example : 17

A and B are the points of intersection of the circles $x^2 + y^2 + 2ax - c^2 = 0$ and $x^2 + y^2 + 2bx - c^2 = 0$. A line through A meets one circle at P. Another line parallel to AP but passing through B cuts the other circle at Q. Find the locus of the mid-point of PQ.

Solution

Let us solve for the point of intersection A and B

$x^2 + y^2 + 2ax - c^2 = 0$ and $x^2 + y^2 + 2bx - c^2 = 0$

$\Rightarrow \qquad x = 0$ and $y = \pm c$

$\Rightarrow \qquad A \equiv (0, c)$ and $B \equiv (0, -c)$

Let the equation of AP be : $y = mx + c$, where m is changing quantity and c is fixed quantity (Y-intercept)

$\Rightarrow$ the equation BQ is : $y = mx - c \qquad (AP \parallel BQ)$

Coordinates of P, Q :

Solve $y = mx + c$ and $x^2 + y^2 + 2ax - c^2 = 0$

$\Rightarrow$ $x^2 (mx + c)^2 + 2ax + c^2 = 0$

$$\Rightarrow \quad x = -\frac{2(a+mc)}{1+m^2} \quad \text{and} \quad x = 0$$

$$\Rightarrow \quad y = -\frac{2m(a+mc)}{1+m^2} + c \quad \text{and} \quad y = c$$

$$\Rightarrow \quad P = \left[\frac{2(a+mc)}{1+m^2}, -\frac{2m(a+mc)}{1+m^2} + c\right]$$

Similarly the coordinates Q are :

$$\Rightarrow \quad Q \equiv \left[-\frac{2(b-mc)}{1+m^2}, -\frac{2m(b-mc)}{1+m^2} - c\right]$$

mid-point of PQ is :

$$\left[-\frac{(a+b)}{1+m^2}, -\frac{m(a+b)}{1+m^2}\right] \equiv (x_1, y_1)$$

$$\Rightarrow \quad x_1 = -\frac{(a+b)}{1+m^2} \; ; \; y_1 = -\frac{m(a+b)}{1+m^2}$$

Elimiate m to get the locus of the midpoint

$x_1^2 + y_1^2 = -(a + b)\, x_1$

$\Rightarrow$ $x^2 + y^2 + (a + b)\, x = 0$ is the locus

Example : 18

Find the equation of the circumcircle of the triangle having $x + y = 6$, $2x + y = 4$ and $x + 2y = 5$ as its sides.

Solution

Consider the following equation :

$(x + y - 6)(2x + y - 4) + \lambda (2x + y - 4)(x + 2y - 5) + \mu (x + 2y - 5)(x + y - 6) = 0$ ....(i)

Equation (i) represents equation of curve passing through the intersection of the three lines taken two at a time (i.e. passes through the vertices of the triangle). For this curve to represent a circle,

Coefficient of x^2 = Coefficient of y^2 and Coefficient of $xy = 0$

$\Rightarrow$ $2 + 2\lambda + \mu = 1 + 2\lambda + 2\mu$ ....(ii)

and $3 + 5\lambda + 3\mu = 0$ ....(iii)

Solving (ii) and (iii), we get $\lambda = -6/5$ and $\mu = 1$

Putting values of λ and μ in (i), we get :

$(x + y - 6)(2x + y - 4) - 6/5\,(2x + y - 4)(x + 2y - 5) + 1\,(x + 2y - 5)(x + y - 6) = 0$

$\Rightarrow$ $x^2 + y^2 - 17x - 19y + 50 = 0$

Hence equation of circumcircle of the triangle is : $x^2 + y^2 - 17x - 19y + 50 = 0$

Example : 19

Find the equation of the circle passing through the origin and through the points of contact of tangents from the origin to the circle $x^2 + y^2 - 11x + 13y + 17 = 0$

Solution

Let $S = x^2 + y^2 - 11x + 13y + 17 = 0$

Equation of the chord of contact of circle S with respect to the point (0, 0) is

$L \equiv -11x + 13y + 34 = 0$

Equation of family of circles passing through the intersection of circle S and chord of contact L is

$S + kL = 0$

$\Rightarrow$ $x^2 + y^2 - 11x + 13y + 17 + k(-11x + 13y + 34) = 0$ ....(i)

Since required circle passes through the origin, find the member of this family that passes through the origin

i.e. Put (0, 0) and find corresponding value of k.

$\Rightarrow$ $0^2 + 0^2 - 11 \times 0 + 13 \times 0 + 17 + k(-11 \times 0 + 13 \times 0 + 34) = 0$

$\Rightarrow$ $k = -1/2$

Put $k = -1/2$ in (i) to get equation of the required circle i.e. $2x^2 + 2y^2 - 11x + 13y = 0$

Alternate Solution

Let centre of the circle S be C. As points of contact, origin and C form a cyclic quadrilateral, OC must be the diameter of the required circle.

$C \equiv (11/2, -13/2)$ and $O \equiv (0, 0)$

Apply diametric form to get the equation of the required circle,

i.e. $(x - 11/2)(x - 0) + (y + 13/2)(y - 0) = 0$

$\Rightarrow \quad 2x^2 + 2y^2 - 11x + 13y = 0$

Hence required circle is : $2x^2 + 2y^2 - 11x + 13y = 0$

Example : 20

If $\left(m_i, \frac{1}{m_i}\right)$, $m_i > 0$ for i = 1, 2, 3, 4 are four distinct points on a circle. Show that $m_1m_2m_3m_4 = 1$.

Solution

Let equation of circle be $x^2 + y^2 + 2gx + 2fy + c = 0$

As $\left(m_i, \frac{1}{m_i}\right)$ lies on the circle, it should satisfy the equation of the circle

i.e. $m_1^2 + \frac{1}{m_i^2} + 2gm_1 + 2f\,\frac{1}{m_i} + c = 0$

$\Rightarrow \quad m_i^4 + 2gm_i^3 + cm_i^2 + 2fm_i + 1 = 0$

This is equation of degree four in m whose roots are m_1, m_2, m_3, and m_4.

Product of the roots = $m_1m_2m_3m_4 = \frac{\text{coefficient of } x^0}{\text{coefficient of } x^4} = \frac{1}{1} = 1$

Hence $m_1m_2m_3m_4 = 1$

Example : 21 $\left(\frac{ma}{1+m^2}, \frac{ma}{1+m^2}\right)$

y = mx is a chord of the circle of radius a and whose diameter is along the axis of x. Find the equation of the circle whose diameter is this chord and hence find the locus of its centre for all values of m.

Solution

The circle whose chord is y = mx and centre lies on x -axis will touch y axis at origin

The equation of such circle is given by :

$(x - a)^2 + y^2 = a^2 \quad \Rightarrow \quad x^2 + y^2 - 2ax = 0$ ....(i)

Further, family of circles passing through the intersection of circle (i) and the line y = mx is :

$x^2 + y^2 - 2ax + k(y - mx) = 0 \quad \Rightarrow \quad x^2 + y^2 - x(2a + km) + ky = 0$ ....(i)

centre of the circle is $\equiv (a + km/2, -k/2)$

We require that member of this family whose diameter is y = mx

$\Rightarrow$ centre of the required circle lies on y = mx.

$\Rightarrow \quad -k/2 = am + km^2/2 \quad \Rightarrow \quad k = -2ma/(1 + m^2)$

Put the value of k in (i) to get the equation of the required circle,

$$x^2 + y^2 - x\left(2a - \frac{2am^2}{1+m^2}\right) - \frac{2am}{1+m^2}\,y = 0$$

$\Rightarrow \quad (1 + m^2) - (x^2 + y^2) - 2a(x + my) = 0$

(ii) Let the coordinates of the point whose locus is required be (x_1, y_1)

$\Rightarrow \quad (x_1, y_1)$ is the centre of the circle (ii)

$\Rightarrow \quad (x_1, y_1) \equiv$

$\Rightarrow \quad x_1 = \frac{a}{1+m^2}$ ....(iii) and $y_1 =$ ....(iv)

On squaring and adding (iii) and (iv), we get :

$x_1^2 + y_1^2 =$ $\Rightarrow$ $1 + m^2 =$

Substitute the value of $(1 + m^2)$ in (iii) to get : $x_1^2 + y_1^2 = ax_1$

$\Rightarrow$ required locus is : $x^2 + y^2 = ax$.

Example : 22

Find the equation of a circle having the lines $x^2 + 2xy + 3x + 6y = 0$ as its normals and having size just sufficient to contain the circle $x (x – 4) + y (y – 3) = 0$

Solution

On factorising the equation of the pair of straight lines $x^2 + 2xy + 3x + 6y = 0$, we get :

$(x + 2y) (x + 3) = 0$

$\Rightarrow$ Two normals are $x = –2y$ ....(i) and $x = – 3$ ....(ii)

The point of intersection of normals (i) and (ii) is centre of the required circle as centre lies on all normal lines.

Solving (i) and (ii), we get :

centre $\equiv C_1 \equiv (–3, 3/2)$

Given circle is $C_2 \equiv$` $x (x – 4) + y(y – 3) = 0$ $\Rightarrow$ $x^2 + y^2 – 4x – 3y = 0$

$\Rightarrow$ centre $\equiv C_2 \equiv (2, 3/2)$ and radius = $r = 5/2$

If the required circle just contains the given circle, the given circle should touch the required circle internally from inside.

$\Rightarrow$ radius of the required circle = $|C_1 – C_2| + r$

$\Rightarrow$ radius of the required circle = $5 + 5/2 = 15/2$

Hence, equation of required circle is $(x + 3)^2 + (y – 3/2)^2 = 225/4$

Example : 23

A variable circle passes through the point (a, b) and touches the x-axis. Show that the locus of the other end of the diameter through A is $(x – a)^2 = 4by$

Solution

Let the equation of the variable circle be $x^2 + y^2 + 2gx + 2fy + c = 0$

Let $B \equiv (x_1, y_1)$ be the other end of the diameter whose locus is required

centre of the circle $\equiv (–g, –f) \equiv$ mid point of the diameter AB $\equiv \left(\frac{x_1 + a}{2}, \frac{y_1 + b}{2}\right)$

$\Rightarrow$ $–2g = x_1 + a$ ....(i) and $–2f = y_1 + b$ ....(ii)

As circle touches x axis, we can write : | f | = radius of the circle

$\Rightarrow$ $| f |^2 = g^2 + f^2 – c$ $\Rightarrow$ $g^2 = c$

Substituting the value of g from (i), we get : $c = (x_1 + a^2)/4$ ....(iii)

Since point $B \equiv (x_1, y_1)$ lies on circle, we can have :

$x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c = 0$

On substituting the values of g, f and c from (i), (ii) and (iii), we get :

$x_1^2 + y_1^2 – (x_1 + a) x_1 – (y_1 + b) y_1 + (x_1 + a)^2 /4 = 0$

$\Rightarrow$ $(x_1 – a)^2 = 4by_1$

Hence, required locus is $(x – a)^2 = 4by$

Alternate Solution

Let $B \equiv (x_1, y_1)$ be the other end of the diameter whose locus is required

centre of the circle $\equiv (–g, –f) \equiv$ mid point of the diameter AB $\equiv \left(\frac{x_1 + a}{2}, \frac{y_1 + b}{2}\right)$

length of the diameter of the circle = $[(x_1 – a)^2 + (y_1 – b)^2]^{1/2}$

$\Rightarrow$ radius = $r = 1/2 [(x_1 – a)^2 + (y_1 – b)^2]^{1/2}$

As circle touches x-axis, $| f | = r$ $\Rightarrow$ $| f |^2 = r^2$

$\Rightarrow$ $(y_1 + b)^2 = (x_1 – a)^2 + (y_1 – b)^2$

$\Rightarrow$ $(x_1 – a)^2 = 2by_1$

Hence, required locus is $(x – a)^2 = 4by$

Example : 24

A circle is drawn so that it touches the y-axis cuts off a constant length 2a, on the axis of x. Show that the equation of the locus of its centre is $x^2 - y^2 = a^2$.

Solution

Let (x_1, y_1) be the centre of the circle.

As circle touches y-axis, radius of the circle = x_1.

So equation of circle is : $(x - x_1)^2 + (y - y_1)^2 = x_1^2$

$\Rightarrow \quad x^2 + y^2 - 2x_1x - 2y_1y + y_1^2 = 0$

Intercept made by the circle on x-axis = $2(g^2 - c)^{1/2} = 2a$ (given)

$\Rightarrow \quad g^2 - c = a^2 \quad \Rightarrow \quad x_1^2 - y_1^2 = a^2$

Hence required locus is $x^2 - y^2 = a^2$

Example : 25

A circle is cut by a family of circles all of which pass through two given points $A \equiv (x_1, y_1)$ and $B(x_2, y_2)$. prove that the chords of intersection of the fixed circle with any circle of the family passes through a fixed point.

Solution

Let $S_0 \equiv 0$ be the equation of the fixed circle.

Equation of family of circles passing through two given points A and B is :

$$S_2 \equiv (x - x_1)(x - x_2) + (y - y_1)(y - y_2) + kL_1 = 0$$

where L_1 is equation of line passing through A and B

$\Rightarrow \quad S_2 \equiv S_1 + kL_1$ ....(i)

where $S_1 \equiv (x - x_1)(x - x_2) + (y - y_1)(y - y_2)$

The common chord of intersecting of circles $S_0 = 0$ and $S_2 = 0$ is given by :

$$L \equiv S_2 - S_0 = 0$$

Using (i), we get

$$L \equiv S_2 - S_1 - kL_1 = 0$$

$\Rightarrow \quad L \equiv L_2 - kL_1$ where $L_2 \equiv S_2 - S_1$ is the equation fo common chord of S_1 and S_2 .

On observation we can see that L represents a family of straight lines passing the intersection of L_2 and L_1. Hence all common chords (represented by L) pass through a fixed point

Example : 26

The circle $x^2 + y^2 = 1$ cuts the x-axis at P and Q. Another circle with centre at Q and variable radius intercepts the first circle at R above x-axis and the line segment PQ at S. Find the maximum area of the triangle QSR

Solution

Equation of circle I is $x^2 + y^2 = 1$. It cuts x-axis at point P (1, 0) and Q(–1, 0).

Let the radius of the variable circle be r. Centre of the variable circle is Q(–1, 0)

$\Rightarrow \quad$ Equation of variable circle is $(x + 1)^2 + y^2 = r^2$ ....(ii)

Solving circle I and variable circle we get coordinates of R as $\left(\dfrac{r^2 - 2}{2}, \dfrac{r}{2}\sqrt{4 - r^2}\right)$

Area of the triangle QSR = 1/2 × QS × RL = $\dfrac{1}{2}\, r\, \dfrac{r}{2}\sqrt{4 - r^2}$

To maximise the area of the triangle, maximise its square i.e.

Let $\quad A(r) = \dfrac{1}{16} r^4 (4 - r^2) = \dfrac{4r^4 - r^6}{16}$

$\Rightarrow \quad A'(r) = \dfrac{16r^3 - 6r^5}{16}$

For A(r) to be maximum or minimum, equate $A'(r) = 0$

$\Rightarrow \quad r = \sqrt{\dfrac{8}{3}}$

See yourself that $A''\left(\sqrt{\dfrac{8}{3}}\right) < 0$

$\Rightarrow$ Area is maximum for $r = \sqrt{\frac{8}{3}}$

Maximum Area of the triangle QRS = $\frac{1}{2} \cdot \frac{1}{2}\ \frac{8}{3} \cdot \sqrt{\frac{4}{3}} = \frac{4}{3\sqrt{3}}$ sq. units.

Example : 27

Two circles each of radius 5 units touch each other at (1, 2). If the equation of their common tangent is 4x + 3y = 10, find the equation of the circles.

Solution

Equation of common tangent is 4x + 3y = 10. The two circles touch each other at (1, 2).

Equation of family of circles touching a given line 4x + 3y = 10 at a given point (1, 2) is :

$$(x - 1)^2 + (y - 2)^2 + k\,(4x + 3y - 10) = 0$$

$\Rightarrow$ $x^2 + y^2 + (4k - 2)\,x + (3k - 4)\,y + 5\ \ 10k = 0$ ......(i)

$\Rightarrow$ centre $\equiv \left(1 - 2k, \frac{4 - 3k}{2}\right)$ and radius = $g^2 + f^2 - c = (2k - 1)^2 + \left(\frac{3k - 4}{2}\right)^2 - (5 - 10k)$

As the radius of the required circle is 5, we get : $(2k - 1)^2 + \left(\frac{3k - 4}{2}\right)^2 - (5 - 10k) = 5$

$\Rightarrow$ $k^2 = 20/25$ $\Rightarrow$ $k = \pm \frac{2}{\sqrt{5}}$

Put the values of k in (i) to get the equations of required circles.

The required circles are : $\sqrt{5}\ (x^2 + y^2) + (8 - 2\sqrt{5})\ x + (6 - 4\sqrt{5})\ y + 5\sqrt{5} - 20 = 0$

and $\sqrt{5}\ (x^2 + y^2) + (8 + 2\sqrt{5})\ x - (6 + 4\sqrt{5})\ y + 5\sqrt{5} + 20 = 0$

Example : 28

The line Ax + By + C = 0 cuts the circle $x^2 + y^2 + ax + by + c = 0$ in P and Q. The line $A'x + B'y + c' = 0$ cuts the circle $x^2 + y^2 + a'x + b'y + c' = 0$ in R and S. If P, Q, R and S are concyclic then show that

$$\begin{vmatrix} a - a' & b - b' & c - c' \\ A & B & C \\ A' & B' & C' \end{vmatrix} = 0$$

Solution

Let the given circles be $S_1 \equiv x^2 + y^2 + ax + by + c = 0$ and $S_2 \equiv x^2 + y^2 + a'x + b'y + c' = 0$. Assume that the points P, Q, R and S lie on circle $S_3 = 0$

The line PQ $\equiv$ Ax + By + C = 0 intersects both S_1 and S_3.

$\Rightarrow$ Line PQ is radical axis of S_1 and S_3

The line RS $\equiv$ $A'x + B'y + c' = 0$ intersects both S_2 and S_3

$\Rightarrow$ Line RS is radical axis of S_2 and S_3 .

Also radical axis of $S_1 = 0$ and $S_2 = 0$ is given by : $S_1 - S_2 = 0$

or $(a - a')x + (b - b')y + c - c' = 0$ ....(i)

The lines PQ, RS and line (i) are concurrent lines because radical axis of three circles taken in pair are concurrent. Using the result of three concurrent lines, we get :

$$\begin{vmatrix} a - a' & b - b' & c - c' \\ A & B & C \\ A' & B' & C' \end{vmatrix} = 0$$

Example : 29

If two curves whose equations are : $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ and

$a'x^2 + 2h'xy + b'y^2 + 2g'x + 2f'y + c' = 0$ intersect in four concyclic points, prove that $\frac{a-b}{h} = \frac{a'-b'}{h'}$.

Solution

The equation of family of curves passing through the points of intersection of two curves is :

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c + k\,(a'x^2 + 2h'xy + b'y^2 + 2g'x + 2f'y + c') = 0$$

It above equation represents a circle, then coefficient of x^2 = coefficient of y^2 and coefficient of xy = 0

$\Rightarrow \quad a + ka' = b + kb' \qquad$ ....(i)

and $\quad 2(h + kh') = 0 \quad \Rightarrow \quad k = -h/h'$

On substituting the value of k in (i), we get :

$$\frac{a-b}{h} = \frac{a'-b'}{h'}$$

Example : 30

Find all the common tangents to the circles $x^2 + y^2 - 2x - 6y + 9 = 0$ and $x^2 + y^2 + 6x - 2y + 1 = 0$.

Solution

The centre and radius of first circle are : $C_1 \equiv (1, 3)$ and $r_1 = 1$

The centre and radius of second circle are : $C_1 \equiv (-3, 1)$ and $r_2 = 3$

Direct common tangents

Let P be the point of intersection of two direct common tangents.

Using the result that divides C_1C_2 externally in the ratio of radii i.e. 1 : 3

the coordinates of point P are $\quad P \equiv \left(\frac{1(-3) - 3\,.\,1}{1-3}, \frac{1.1 - 3(3)}{1-3}\right) \equiv (3, 4)$

Let m be the slope of direct common tangent.

So equation of direct common tangent is : $y - 4 = m\,(x - 3)$ ....(i)

Since direct common tangent touches circles, apply condition of tangency with first circle

i.e. $\quad \frac{|-1+2m|}{\sqrt{1+m^2}} = 1 \quad \Rightarrow \quad 1 = 4m^2 - 4m = 1 + m^2$

$\Rightarrow \quad 3m^2 + 4m = 0 \quad \Rightarrow \quad m\,(3m + 4) = 0$

$\Rightarrow \quad m = 0$ and $m = 4/3$

On substituting the values of m in (i), we get the equations of two direct common tangents

i.e. $\quad y = 4$ and $4x - 3y = 0$

Hence equations of direct common tangents are : $y = 4$ and $4x - 3y = 0$

<u>**Transverse common tangents**</u>

Let Q be the point of intersection fo two transverse (indirect) common tangents.

Using the result that P divides C_1C_2 internally in the ratio radii i.e. 1 : 3

the coordinates of point P are $P \equiv \left(\frac{1(-3) + 3.1}{1+3}, \frac{1.1 + 3(3)}{1+3}\right) \equiv \left(0, \frac{5}{2}\right)$

Let m be the slope of direct common tangent.

So equation of direct common tangent is : $y - 5/2 = mx$ ....(i)

Since direct common tangent touches circles, apply condition of tangency with first circle

i.e. $\quad \frac{|m - 1/2|}{\sqrt{1+m^2}} = 1 \quad \Rightarrow \quad 1 + 4m^2 - 4m = 4 + 4m^2$

$\Rightarrow \quad 0m^2 + 4m + 3 = 0$

As coefficient of m^2 is 0, one root must be ∞ and other is $m = -3/4$

$\Rightarrow \quad m = \infty$ and $m = -3/4$

On substituting the values of m in (i), we get the equations of two direct common tangents

i.e. $\quad x = 0$ and $3x + 4y = 10$

Hence equations of direct common tangents are : $x = 0$ and $3x + 4y = 10$.

Example : 31

Find the intervals of values of a for which the line y + x = 0 bisects two chords drawn from a point $\left(\frac{1+\sqrt{2}a}{2}, \frac{1-\sqrt{2}a}{2}\right)$ to the circle $2x^2 + 2y^2 - (1 + \sqrt{2}a)x - (1 - \sqrt{2}a)y = 0$.

Solution

Let $(m, n) \equiv \left(\frac{1+\sqrt{2}a}{2}, \frac{1-\sqrt{2}a}{2}\right)$

$\Rightarrow$ Equation fo circle reduces to $x^2 + y^2 - mx - ny = 0$.

Let P (t, –t) be a point on the line y + x = 0.

Equation fo chord passing through (t, –t) as mid-point is :

$$xt - yt + \frac{-m}{2}(x + t) + \frac{-n}{2}(y - t) = t^2 + t^2 - mt + nt \qquad \text{...........(i)}$$

Since chord (i) also passes through (m, n), it should satisfy the equation of chord

i.e. $mt - nt + \frac{-m}{2}(m + t) + \frac{-n}{2}(n - t) = t^2 + t^2 - mt + nt$

$\Rightarrow$ $4t^2 + m^2 + n^2 = 3t(m - n)$

On substituting the values of m and n, we get $\Rightarrow$ $4t^2 - 3\sqrt{2}at + (1 + 2a^2)/2 = 0$ ....(ii)

Now if there exists two chords passing through (m, n) and are bisected by the line y + x = 0, then equation of (ii) should have two real and distinct roots.

$\Rightarrow$ $D > 0$ $\Rightarrow$ $18a^2 - 16(1 + 2a^2)/2 > 0$

$\Rightarrow$ $a^2 - 4 > 0$ $\Rightarrow$ $(a + 2)(a - 2) > 0$

$\Rightarrow$ $a \in (-\infty, -2) \cup (2, \infty)$

Hence values of a are $a \in (-\infty, -2) \cup (2, \infty)$.