

Example : 1

Express the following complex numbers in the trigonometric forms and hence calculate their principal arguments. Show the complex numbers on the Argand plane

(i) $z_1 = -\sqrt{3} + i$ (ii) $z_2 = -1 - \sqrt{3}i$ (iii) $z_3 = 1 - i$

Solution

(i) $z_1 = -\sqrt{3} + i$ ($|z| = 2$)

$$\Rightarrow z_1 = 2 \left(-\frac{\sqrt{2}}{3} + \frac{1}{2}i \right) \quad \left(\text{as } \cos \theta = -\frac{\sqrt{3}}{2}, \sin \theta = \frac{1}{2} \right)$$

$$\Rightarrow z_1 = 2 \left(\cos \frac{5\pi}{6} + i \sin \frac{5\pi}{6} \right) \quad \Rightarrow \quad \text{the argument} = \frac{5\pi}{6}$$

(ii) $z_3 = -1 - \sqrt{3}i$ ($|z| = 2$)

$$\Rightarrow z_2 = 2 \left(-\frac{1}{2} - \frac{\sqrt{3}}{2}i \right) \quad \left(\cos \theta = -\frac{1}{2}, \sin \theta = -\frac{\sqrt{3}}{2} \right)$$

$$\Rightarrow z_2 = 2 \left[\cos \left(\frac{-2\pi}{3} \right) + i \sin \left(\frac{-2\pi}{3} \right) \right]$$

$$\Rightarrow \text{argument} = \frac{-2\pi}{3}$$

(iii) $z_3 = 1 - i$ ($|z| = \sqrt{2}$)

$$\Rightarrow z_3 = \sqrt{2} \left(\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i \right) \quad \left(\cos \theta = \frac{1}{\sqrt{2}}, \sin \theta = -\frac{1}{\sqrt{2}} \right)$$

$$\Rightarrow z_3 = \sqrt{2} \left[\cos \left(\frac{-\pi}{4} \right) + i \sin \left(\frac{-\pi}{4} \right) \right]$$

$$\Rightarrow \text{argument} = \frac{-\pi}{4}$$

Example : 2

If $z_1 = r_1 (\cos \alpha + i \sin \alpha)$ and $z_2 = r_2 (\cos \beta + i \sin \beta)$, show that :

(i) $|z_1 z_2| = r_1 r_2$ (ii) $\arg (z_1 z_2) = \alpha + \beta$

(iii) $\left| \frac{z_1}{z_2} \right| = \frac{r_1}{r_2}$ (iv) $\arg \left(\frac{z_1}{z_2} \right) = \alpha - \beta$

Solution

For (i) and (ii) :

$$\begin{aligned} z_1 z_2 &= r_1 r_2 (\cos \alpha + i \sin \alpha) (\cos \beta + i \sin \beta) \\ &= r_1 r_2 (\cos \alpha \cos \beta - \sin \alpha \sin \beta + i \sin \alpha \cos \beta + i \cos \alpha \sin \beta) \\ &= r_1 r_2 [\cos (\alpha + \beta) + i \sin (\alpha + \beta)] \end{aligned}$$

comparing with $z = |z| (\cos \theta + i \sin \theta)$, we get :

$$|z_1 z_2| = r_1 r_2 \quad \text{and} \quad \arg (z_1 z_2) = \alpha + \beta$$

For (iii) and (iv) :

$$\begin{aligned} \frac{z_1}{z_2} &= \frac{r_1 (\cos \alpha + i \sin \alpha)}{r_2 (\cos \beta + i \sin \beta)} \\ &= \frac{r_1}{r_2} (\cos \alpha + i \sin \alpha) (\cos \beta - i \sin \beta) \end{aligned}$$

$$= \frac{r_1}{r_2} [\cos \alpha \cos \beta + \sin \alpha \sin \beta + i \sin \alpha \cos \beta - i \cos \alpha \sin \beta]$$

$$= \frac{r_1}{r_2} [\cos (\alpha - \beta) + i \sin (\alpha + \beta)]$$

$$\Rightarrow \left| \frac{z_1}{z_2} \right| = \frac{r_1}{r_2} \quad \text{and} \quad \arg \left(\frac{z_1}{z_2} \right) = \alpha - \beta$$

Example : 3

Show that $|z - 2i| = 2\sqrt{2}$, if $\arg \left(\frac{z-2}{z+2} \right) = \frac{\pi}{4}$

Solution

Let $z = x + yi$ $x, y \in \mathbb{R}$

$$\Rightarrow \arg \left(\frac{x-2+yi}{x+2+yi} \right) = \frac{\pi}{4}$$

$$\Rightarrow \arg \left[\frac{(x-2+yi)(x+2-yi)}{(x+2)^2 + y^2} \right] = \frac{\pi}{4}$$

$$\Rightarrow \arg \left[\frac{(x^2 - 4 + y^2) + 4yi}{(x+2)^2 + y^2} \right] = \frac{\pi}{4}$$

$$\Rightarrow \frac{4y}{x^2 - 4 + y^2} = \tan \frac{\pi}{4}$$

$$\Rightarrow x^2 + y^2 - 4y - 4 = 0$$

$$\Rightarrow x^2 + (y-2)^2 = 8$$

$$\Rightarrow |x + (y-2)i| = 2\sqrt{2}$$

$$\Rightarrow |z - 2i| = 2\sqrt{2}$$

Example : 4

If $\cos \alpha + \cos \beta + \cos \gamma = \sin \alpha + \sin \beta + \sin \gamma = 0$, then show that :

- (i) $\cos 3\alpha + \cos 3\beta + \cos 3\gamma = 3 \cos (\alpha + \beta + \gamma)$
- (ii) $\sin 3\alpha + \sin 3\beta + \sin 3\gamma = 3 \sin (\alpha + \beta + \gamma)$
- (iii) $\cos 2\alpha + \cos 2\beta + \cos 2\gamma = \sin 2\alpha + \sin 2\beta + \sin 2\gamma = 0$

Solution

For (i) and (ii) :

$$\text{Let } z_1 = \cos \alpha + i \sin \alpha \quad ;$$

$$z_2 = \cos \beta + i \sin \beta \quad ;$$

$$z_3 = \cos \gamma + i \sin \gamma$$

$$z_1 + z_2 + z_3 = \sum \cos \alpha + i \sum \sin \alpha = 0$$

for $3\alpha, 3\beta, 3\gamma$ we have to consider z_1^3, z_2^3, z_3^3

$$z_1^3 + z_2^3 + z_3^3 = (\cos \alpha + i \sin \alpha)^3 + (\cos \beta + i \sin \beta)^3 + (\cos \gamma + i \sin \gamma)^3$$

$$= (\cos 3\alpha + i \sin 3\alpha) + (\cos 3\beta + i \sin 3\beta) + (\cos 3\gamma + i \sin 3\gamma)$$

$$= (\cos 3\alpha + \cos 3\beta + \cos 3\gamma) + i (\sin 3\alpha + \sin 3\beta + \sin 3\gamma) \quad \dots\dots\dots(i)$$

Now $z_1^3 + z_2^3 + z_3^3 = 3z_1 z_2 z_3$ because $z_1 + z_2 + z_3 = 0$

$$\Rightarrow z_1^3 + z_2^3 + z_3^3 = 3 (\cos \alpha + i \sin \alpha) (\cos \beta + i \sin \beta) (\cos \gamma + i \sin \gamma)$$

$$z_1^3 + z_2^3 + z_3^3 = 3 [\cos (\alpha + \beta + \gamma) + i \sin (\alpha + \beta + \gamma)] \quad \dots\dots\dots(ii)$$

Equating the RHS of (i) and (ii), we get :

$$\sum \cos 3\alpha + i \sum \sin 3\alpha = 3 \cos (\alpha + \beta + \gamma) + 3 i \sin (\alpha + \beta + \gamma)$$

Equating real and imaginary parts,

$$\sum \cos 3\alpha = 3 \cos (\alpha + \beta + \gamma) \quad \text{and} \quad \sum \sin 3\alpha = 3 \sin (\alpha + \beta + \gamma)$$

For (iii) :

Consider $z_1^2 + z_2^2 + z_3^2 = (z_1 + z_2 + z_3)^2 - 2(z_1 z_2 + z_2 z_3 + z_3 z_1)$

$$= 0 - 2z_1 z_2 z_3 \left(\frac{1}{z_1} + \frac{1}{z_2} + \frac{1}{z_3} \right)$$

$$= 2z_1 z_2 z_3 \left[\frac{1}{\cos \alpha + i \sin \alpha} + \frac{1}{\cos \beta + i \sin \beta} + \frac{1}{\cos \gamma + i \sin \gamma} \right]$$

$$= -2z_1 z_2 z_3 [\cos \alpha - i \sin \alpha + \cos \beta - i \sin \beta + \cos \gamma - i \sin \gamma]$$

$$= -2z_1 z_2 z_3 [\sum \cos \alpha - i \sum \sin \alpha]$$

$$= -2z_1 z_2 z_3 [0 - i(0)] = 0$$

$$\Rightarrow (\cos \alpha + i \sin \alpha)^2 + (\cos \beta + i \sin \beta)^2 + (\cos \gamma + i \sin \gamma)^2 = 0$$

$$\Rightarrow (\cos \alpha + i \sin 2\alpha) + (\cos 2\beta + i \sin 2\beta) + (\cos 2\gamma + i \sin 2\gamma) = 0$$

$$\Rightarrow \sum \cos 2\alpha = 0 \quad \text{and} \quad \sum \sin 2\alpha = 0$$

Example : 5

Express $\sin 5\theta$ in terms of $\sin \theta$ and hence show that $\sin 36^\circ$ is a root of the equation $16x^4 + 20x^2 + 5 = 0$.

Solution

Expand $(\cos \theta + i \sin \theta)^5$ using binomial theorem.

$$(\cos \theta + i \sin \theta)^5 = {}^5C_0 \cos^5 \theta + 5 {}^5C_1 \cos^4 \theta (i \sin \theta) + \dots + 5 {}^5C_5 i^5 \sin^5 \theta$$

using DeMoivre's theorem on L.H.S. :

$$(\cos 5\theta + i \sin 5\theta) = (\cos^5 \theta - 10 \cos^3 \theta \sin^2 \theta + 5 \cos \theta \sin^4 \theta) + i 5 [\cos^4 \theta \sin \theta - 10 \cos^2 \theta \sin^3 \theta + \sin^5 \theta]$$

Equating imaginary parts :

$$\sin 5\theta = \sin \theta [5 \cos^4 \theta - 10 \cos^2 \theta \sin^2 \theta + \sin^4 \theta]$$

$$\sin 5\theta = \sin \theta [5(1 + \sin^4 \theta - 2 \sin^2 \theta) - 10(1 - \sin^2 \theta) \sin^2 \theta] + \sin^4 \theta$$

$$\sin 5\theta = 16 \sin^5 \theta - 20 \sin^3 \theta + 5 \sin \theta$$

$$\text{for } \theta = 36^\circ, \quad \sin 5\theta = \sin 180^\circ = 0$$

$$\Rightarrow 16 \sin^5 36^\circ - 20 \sin^3 36^\circ + 5 \sin 36^\circ = 0$$

$$\Rightarrow \sin 36^\circ \text{ is a root of } 16x^5 - 20x^3 + 5x = 0$$

$$\text{i.e. } 16x^4 - 20x^2 + 5 = 0$$

Example : 6

If $(1 + x)^n = P_0 + P_1 x + P_2 x^2 + \dots + P_n x^n$, then show that

$$(a) \quad P_0 - P_2 + P_4 - \dots = 2^{n/2} \cos(n\pi/4)$$

$$(b) \quad P_1 - P_3 + P_5 - \dots = 2^{n/2} \sin(n\pi/4)$$

Solution

Consider the identity

$$(1 + x)^n = P_0 + P_1 x + P_2 x^2 + P_3 x^3 + \dots + P_n x^n$$

Put $x = i$ on both the sides

$$(1 + i)^n = P_0 + P_1 i + P_2 i^2 + P_3 i^3 + \dots + P_n i^n$$

$$\left[\sqrt{2} \left(\cos \frac{\pi}{4} + i \sin \frac{\pi}{4} \right) \right]^n = (P_0 - P_2 + P_4 + \dots) + i (P_1 - P_3 + P_5 + \dots)$$

$$2^{n/2} \left(\cos \frac{n\pi}{4} + i \sin \frac{n\pi}{4} \right) = (P_0 - P_2 + P_4 + \dots) + i (P_1 - P_3 + P_5 + \dots)$$

equate the real and imaginary parts.

$$P_0 - P_2 + P_4 - P_6 + \dots = 2^{n/2} \cos \frac{n\pi}{4}$$

$$P_1 - P_3 + P_5 - \dots = 2^{n/2} \sin \frac{n\pi}{4}$$

Example : 7

If a, b, c and d are the roots of the equation $x^4 + P_1x^3 + P_2x^2 + P_3x + P_4 = 0$, then show that :
 $(1 + a^2) (1 + b^2) (1 + c^2) (1 + d^2) = (1 - P_2 + P_4)^2 + (P_3 - P_1)^2$

Solution

As a, b, c and d are the roots of the given equation :

$\Rightarrow (x - a), (x - b), (x - c)$ and $(x - d)$ are the factors of LHS

$\Rightarrow x^4 + P_1x^3 + P_2x^2 + P_3x + P_4 = (x - a) (x - b) (x - c) (x - d)$ is an identity(i)

Put $x = i$ on both sides :

$$i^4 + P_1i^3 + P_2i^2 + P_3i + P_4 = (i - a) (i - b) (i - c) (i - d)$$

$$(1 - P_2 + P_4) + i (P_3 - P_1) = (i - a) (i - b) (i - c) (i - d) \quad \text{.....(ii)}$$

Put $x = -i$ in (i) :

$$i^4 - P_1i^3 + P_2i^2 - P_3i + P_4 = (-i - a) (i - b) (-i - c) (-i - d)$$

$$(1 - P_2 + P_4) - i (P_3 - P_1) = (-i - a) (-i - b) (-i - c) (-i - d) \quad \text{.....(iii)}$$

multiply (ii) and (iii) to get

$$(1 - P_2 + P_4)^2 + (P_3 - P_1)^2 = (1 + a^2) (1 + b^2) (1 + c^2) (1 + d^2)$$

Example : 8

Show that $|z_1 \pm z_2|^2 = |z_1|^2 + |z_2|^2 \pm 2 \operatorname{Re} (z_1 \bar{z}_2)$.

Solution

$$|z_1 \pm z_2|^2 = (z_1 \pm z_2) (\bar{z}_1 \pm \bar{z}_2)$$

$$= z_1 \bar{z}_2 + z_2 \bar{z}_1 \pm (z_1 \bar{z}_2 + \bar{z}_1 z_2)$$

$$= |z_1|^2 + |z_2|^2 \pm (z_1 \bar{z}_2 + \bar{z}_1 z_2)$$

$$= |z_1|^2 + |z_2|^2 \pm 2 \operatorname{Re} (z_1 \bar{z}_2) \quad \text{because } z + \bar{z} = 2 \operatorname{Re} (z)$$

Example : 9

If 1, ω , ω^2 are cube roots of unity. Show that :

$$(1 - \omega + \omega^2) (1 - \omega^2 + \omega^4) (1 - \omega^4 + \omega^8) \dots \dots \dots 2n \text{ factors} = 2^{2n}$$

Solution

$$\text{LHS} = (1 - \omega + \omega^2) (1 - \omega^2 + \omega^4) (1 - \omega^4 + \omega^8) \dots \dots \dots 2n \text{ factors}$$

using $\omega^4 = \omega^{16} = \dots = \omega$ and $\omega^8 = \omega^{32} = \dots = \omega^2$

$$\text{L.H.S.} = (1 - \omega + \omega^2) (1 - \omega^2 + \omega) (1 - \omega + \omega^2) (1 - \omega^2 + \omega) \dots \dots \dots 2n \text{ factors.}$$

$$\text{L.H.S.} = [(1 - \omega + \omega^2) (1 - \omega^2 + \omega)]^n = [(-2\omega) (-2\omega^2)]^n$$

$$\text{L.H.S.} = 2^{2n} = \text{R.H.S.}$$

Example : 10

Prove that the area of the triangle whose vertices are the points z_1, z_2, z_3 on the argand diagram is :

$$\sum \left[\frac{(z_2 - z_3) |z_1|^2}{4iz_1} \right]$$

Solution

Let the vertices of the triangle be

$$A (x_1, y_1) \quad : \quad z_1 = x_1 + iy_1$$

$$B (x_2, y_2) \quad : \quad z_2 = x_2 + iy_2$$

$$C (x_3, y_3) \quad : \quad z_3 = x_3 + iy_3$$

Area of triangle ABC is :

$$\Delta = \frac{1}{2} \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

We have to express the area in terms of z_1, z_2 and z_3 .

Operating $C_1 \rightarrow C_1 + iC_2$ (properties of Determinants)

$$\Delta = \frac{1}{2} \begin{vmatrix} x_1 + iy_1 & y_1 & 1 \\ x_2 + iy_2 & y_2 & 1 \\ x_3 + iy_3 & y_3 & 1 \end{vmatrix}$$

$$\Delta = \frac{1}{2} \begin{vmatrix} z_1 & y_1 & 1 \\ z_2 & y_2 & 1 \\ z_3 & y_3 & 1 \end{vmatrix}$$

$$\Delta = \frac{1}{4i} \begin{vmatrix} z_1 & z_1 - \bar{z}_1 & 1 \\ z_2 & z_2 - \bar{z}_2 & 1 \\ z_3 & z_3 - \bar{z}_3 & 1 \end{vmatrix}$$

Operating $C_2 \rightarrow C_2 - C_1$ (properties of Determinants)

$$\Delta = \frac{1}{4i} \begin{vmatrix} z_1 & \bar{z}_1 & 1 \\ z_2 & \bar{z}_2 & 1 \\ z_3 & z_3 & 1 \end{vmatrix}$$

$$\Rightarrow \frac{1}{4i} [\bar{z}_1 (z_2 - z_3) + \bar{z}_2 (z_1 - z_3) - \bar{z}_3 (z_1 - z_2)]$$

$$\Rightarrow \Delta = \frac{1}{4i} [\bar{z}_1 (z_2 - z_3) + \bar{z}_2 (z_3 - z_1) - \bar{z}_3 (z_1 - z_2)]$$

$$\Rightarrow \Delta = \frac{1}{4i} \sum \bar{z}_1 (z_2 - z_3)$$

$$\Rightarrow \Delta = \frac{1}{4i} \sum \left[\frac{|\bar{z}_1|^2 (z_2 - z_3)}{z_1} \right]$$

Example : 11

Show that the sum of nth roots of unity is zero.

Solution

Let $S = 1 + e^{i2\pi/n} + e^{i4\pi/n} + \dots + e^{i2\pi(n-1)/n}$
the series on the RHS is a GP

$$\Rightarrow S = \frac{1 \left(1 - e^{i\frac{2\pi}{n}n} \right)}{1 - e^{i\frac{2\pi}{n}}} \Rightarrow S = \frac{1 - e^{i2\pi}}{1 - e^{i\frac{2\pi}{n}}}$$

$$\Rightarrow S = \frac{1 - 1}{1 - e^{i\frac{2\pi}{n}}} = 0$$

Example : 12

Find the value of : $\sum_{r=1}^{r=6} \left[\sin \frac{2\pi r}{7} - i \cos \frac{2\pi r}{7} \right]$

Solution

$$\text{Let } S = \sum_{r=1}^{r=6} \left[\sin \frac{2\pi r}{7} - i \cos \frac{2\pi r}{7} \right] = -i \sum_{r=1}^{r=6} \left[\cos \frac{2\pi r}{7} + i \sin \frac{2\pi r}{7} \right]$$

$$\begin{aligned}
&= -i \sum_{r=1}^{r=6} e^{i\frac{2\pi r}{7}} = -i \left[\sum_{r=0}^{r=6} e^{i\frac{2\pi r}{7}} - 1 \right] \\
&= -i (\text{sum of 7th roots of unity} - 1) \\
&= -i(0 - 1) = i
\end{aligned}$$

Example : 13

Find the sixth roots of $z = i$

Solution

$$z = 1 \left(\cos \frac{\pi}{2} + i \sin \frac{\pi}{2} \right)$$

$$z^{1/6} = 1^{1/6} \left(\cos \frac{\pi/2 + 2k\pi}{6} + i \sin \frac{\pi/2 + 2k\pi}{6} \right) \quad \text{where } k = 0, 1, 2, 3, 4, 5$$

⇒ The sixth roots are :

$$k = 0 \quad \Rightarrow \quad z_n = \left(\frac{\pi}{12} + i \sin \frac{\pi}{12} \right)$$

$$k = 1 \quad \Rightarrow \quad z_1 = \cos \frac{5\pi}{12} + i \sin \frac{5\pi}{12}$$

$$k = 2 \quad \Rightarrow \quad z_2 = \cos \frac{9\pi}{12} + i \sin \frac{9\pi}{12}$$

$$k = 3 \quad \Rightarrow \quad z_3 = \cos \frac{13\pi}{12} + i \sin \frac{13\pi}{12} = \cos \frac{11\pi}{12} - i \sin \frac{11\pi}{12}$$

$$k = 4 \quad \Rightarrow \quad z_4 = \cos \frac{17\pi}{12} + i \sin \frac{17\pi}{12} = -\cos \frac{5\pi}{12} + i \sin \frac{5\pi}{12}$$

$$k = 5 \quad \Rightarrow \quad z_5 = \cos \frac{21\pi}{12} + i \sin \frac{21\pi}{12} = \cos \frac{3\pi}{12} - i \sin \frac{3\pi}{12}$$

Example : 14

Prove that $(x + y)^n - x^n - y^n$ is divisible by $xy(x + y)(x^2 + y^2 + xy)$ if n is odd but not a multiple of 3.

Solution

Let $f(x) = (x + y)^n - x^n - y^n$

$f(0) = (0 + y)^n - (0)^n - y^n = 0$

⇒ $(x - 0)$ is a factor of $f(x)$

⇒ x is a factor of $f(x)$

By symmetry y is also a factor of $f(x)$

$f(-y) = (-y + y)^n - (-y)^n - y^n = 0$ (because n is odd)

⇒ $(x + y)$ is also a factor of $f(x)$.

Now consider $f(\omega y)$

$f(\omega y) = (\omega y + y)^n - (\omega y)^n - y^n$

$= y^n (-\omega^2)^n - \omega^n y^n - y^n$

$= y^n [-\omega^{2n} - \omega^n - 1]$ (because n is odd)

$= -y^n [\omega^{2n} + \omega^n + 1]$

n is not a multiple of 3.

⇒ $n = 3k + 1$ or $n = 3k + 2$ where k is an integer

⇒ $[\omega^{2n} + \omega^n + 1] = 0$ (for both cases)

⇒ $f(\omega y) = 0$

⇒ $(x - \omega y)$ is also a factor of $f(x)$

Similarly we can show that $f(\omega^2 y) = 0$

⇒ $(x - \omega^2 y)$ is also a factor of $f(x)$

Combining all the factors :

we get : $xy(x+y)(x-\omega^2y)(x-\omega^2y)$ is a factor of $f(x)$
 now $(x-\omega y)(x-\omega^2y) = x^2 + xy + y^2$
 $\Rightarrow f(x)$ is divisible by $xy(x+y)(x-\omega y)(x-\omega^2y)$

Example : 15

Interpret the following equations geometrically on the Argand plane :

(i) $|z - 2 - 3i| = 4$ (ii) $|z - 1| + |z + 1| = 4$

(iii) $\arg\left(\frac{z-1}{z+1}\right) = \frac{\pi}{4}$ (iv) $\frac{\pi}{6} < \arg(z) < \frac{\pi}{3}$

Solution

To interpret the equations geometrically, we will convert them to Cartesian form in terms of x and y coordinates by substituting $z = x + iy$

(i) $|x + iy - 2 - 3i| = 4$
 $\Rightarrow (x-2)^2 + (y-3)^2 = 4^2$
 \Rightarrow the equation represents a circle centred at $(2, 3)$ of radius 4 units

(ii) $|x + iy - 1| = |x + iy + 1| = 4$
 $\Rightarrow \sqrt{(x-1)^2 + y^2} + \sqrt{(x+1)^2 + y^2} = 4$

simplify to get : $\frac{x^2}{4} + \frac{y^2}{3} = 1$

\Rightarrow the equation represents an ellipse centred at $(0, 0)$

(iii) $\text{Arg}\left(\frac{x+iy-1}{x+iy+1}\right) = \frac{\pi}{4}$

$\Rightarrow \text{Arg}(x+iy-1) - \text{Arg}(x+iy+1) = \frac{\pi}{4}$

$\Rightarrow \frac{\frac{y}{x-1} - \frac{y}{x+1}}{1 + \frac{y^2}{x^2-1}} = \tan \frac{\pi}{4} \Rightarrow \frac{2y}{x^2+y^2-1} = 1$

$\Rightarrow x^2 + y^2 - 2y - 1 = 0$

\Rightarrow the equation represents a circle centred at $z = 0 + i$ and of radius $= \sqrt{2}$.

(iv) $\frac{\pi}{6} < \tan^{-1}\left(\frac{y}{x}\right) < \frac{\pi}{3}$

$\Rightarrow \frac{1}{\sqrt{3}} x < y < \sqrt{3} x$

\Rightarrow this inequation represents the region between the lines :
 $y = \sqrt{3} x$ and $y = (1/\sqrt{3}) x$ in Q_1

Example : 16

Find the complex number having least positive argument and satisfying $|z - 5i| \leq 3$

Solution

We will analyse the problem geometrically.

All complex numbers (z) satisfying $|z - 5i| \leq 3$ lies on or inside the circle of radius 3 centred at $z_0 = 5i$.

The complex number having least positive argument in this region is at the point of contact of a tangent drawn from origin to the circle.

From triangle OAC

$OA = \sqrt{5^2 - 3^2} = 4$

$$\text{and } \theta_{\min} = \sin^{-1} \left(\frac{OA}{OC} \right) = \sin^{-1} \left(\frac{4}{5} \right)$$

the complex number at A has modulus 4 and argument $\sin^{-1} 4/5$

$$\Rightarrow z_A = 4 (\cos \theta + i \sin \theta) = 4 \left(\frac{3}{5} + i \frac{4}{5} \right)$$

$$\Rightarrow z_A = \frac{12}{5} + i \frac{16}{5}$$

Example : 17

Show that the area of the triangle on the Argand plane formed by the complex numbers z , iz and $(z + iz)$ is $(1/2) |z|^2$.

Solution

$$iz = ze^{i\pi/2}$$

$\Rightarrow iz$ is the vector obtained by rotating z in anti-clockwise direction through 90°

As $|iz| = |i| |z|$, the triangle is an isosceles right angled triangle.

$$\text{Area} = 1/2 = \text{base} \times \text{height} = 1/2 |z| |iz|$$

Example : 18

If $|z|^2 = 5$, find the area of the triangle formed by the complex numbers z , ωz and $z + \omega z$ as its sides.

Solution

$$\omega z = ze^{i2\pi/3} \quad \text{and} \quad |\omega z| = |z|$$

$\Rightarrow \omega z$ is the vector obtained by rotating vector z anti-clockwise through an angle of 120°

As seen from the figure, the triangle formed is equilateral because angle between equal sides is 60°

$$\Rightarrow \text{Area} = \sqrt{3}/4 (\text{side})^2 = \sqrt{3}/4 |z|^2 = \sqrt{3} \text{ sq. units.}$$

Note that the third side is

$$z + \omega z = (1 + \omega) z = -\omega^2 z = e^{i\pi} e^{-i2\pi/3} z = z e^{i\pi/3}$$

\Rightarrow this vector is obtained by rotating the vector z anticlockwise through 60° . This can be verified from the figure

Example : 19

Show that z_1, z_2, z_3 represent the vertices of an equilateral triangle if and only if :

$$z_1^2 + z_2^2 + z_3^2 - z_1 z_2 - z_2 z_3 - z_3 z_1 = 0$$

Solution

The problem has two parts :

- (i) If the triangle is equilateral then prove the condition
- (ii) If the condition is given then prove the triangle is equilateral.

Part (i)

If the triangle ABC is equilateral, the vector BC can be obtained by rotating AB anti-clockwise through 120°

$$\Rightarrow (z_3 - z_2) = (z_2 - z_1) e^{i2\pi/3}$$

$$\Rightarrow z_3 - z_2 = (z_2 - z_1) \omega$$

$$\Rightarrow z_1 \omega - z_2 \omega - z_2 + z_3 = 0$$

$$\Rightarrow z_1 - z_2 \omega^3 - z_2 \omega^2 + z_3 \omega^2 = 0$$

$$\Rightarrow z_1 - (1 + \omega^2) z_2 + \omega^2 z_3 = 0$$

$$\Rightarrow z_1 + \omega z_2 + \omega^2 z_3 = 0$$

Taking LHS :

$$z_1^2 + z_2^2 + z_3^2 - z_1 z_2 - z_2 z_3 - z_3 z_1 = (z_1 + \omega z_2 + \omega^2 z_3) (z_1 + \omega^2 z_2 + \omega z_3) = 0 \quad (\text{using the above proved result})$$

Part (ii)

Give that :

$$z_1^2 + z_2^2 + z_3^2 - z_1 z_2 - z_2 z_3 - z_3 z_1 = 0$$

$$\Rightarrow (z_1 + \omega z_2 + \omega^2 z_3) (z_1 + \omega^2 z_2 + \omega z_3) = 0$$

$$\Rightarrow (z_1 + \omega z_2 + \omega^2 z_3 = 0 \quad \text{OR} \quad (z_1 + \omega^2 z_2 + \omega z_3) = 0$$

Case (1) :

$$(z_1 + \omega z_2 + \omega^2 z_3) = 0$$

$$\Rightarrow z_1 + \omega z_2 + (-1 - \omega) z_3 = 0$$

- ⇒ $(z_1 - z_3) = \omega (z_3 - z_2)$
- ⇒ $(z_1 - z_2)$ is obtained by rotating the vector $(z_3 - z_2)$ anti-clockwise through 120°
- ⇒ $|z_1 - z_3| = |z_3 - z_2|$ and the angle inside the triangle is 60°
- ⇒ triangle ABC is equilateral

Case (2) :

- $(z_1 + \omega^2 z_2 + \omega z_3) = 0$
- ⇒ $z_1 + \omega z_3 + (-1 - \omega) z_2 = 0$
- ⇒ $(z_1 - z_2) = \omega (z_2 - z_3)$
- ⇒ $|z_1 - z_2|$ is obtained by rotating the vector $(z_3 - z_2)$ anti-clockwise through 120°
- ⇒ $|z_1 - z_2| = |z_2 - z_3|$ and the angle inside the triangle is 60°
- ⇒ triangle ABC is equilateral

Example : 20

Let the complex numbers z_1, z_2 and z_3 be the vertices of an equilateral triangle. Let z_0 be the circumcentre of the triangle. Prove that : $z_1^2 + z_2^2 + z_3^2 = 3z_0^2$.

Solution

For an equilateral triangle with vertices z_1, z_2 and z_3 :

$$z_1^2 + z_2^2 + z_3^2 - z_1 z_2 - z_2 z_3 - z_3 z_1 = 0 \quad \dots\dots\dots(i)$$

As circumcentre coincides with centroid, z_0 is centroid also.

$$\Rightarrow z_0 = (z_1 + z_2 + z_3)/3$$

$$\Rightarrow 9z_0^2 = z_1^2 + z_2^2 + z_3^2 + 2(z_1 z_2 + z_2 z_3 + z_3 z_1)$$

using (i), we have

$$\Rightarrow 9z_0^2 = z_1^2 + z_2^2 + 2(z_1^2 + z_2^2 + z_3^2)$$

$$\Rightarrow 9z_0^2 = 3(z_1^2 + z_2^2 + z_3^2)$$

$$\Rightarrow 3z_0^2 = z_1^2 + z_2^2 + z_3^2$$

Example : 21

If $z_1^2 + z_2^2 - 2z_1 z_2 \cos \theta = 0$, then the origin, z_1, z_2 from vertices of an isosceles triangle with vertical angle θ .

Solution

$$z_1^2 + z_2^2 - 2z_1 z_2 \cos \theta = 0$$

$$\Rightarrow z_1^2 - (2z_2 \cos \theta) z_1 + z_2^2 = 0$$

Solving as a quadratic in z_1 , we get :

$$z_1 = \frac{2z_2 \cos \theta \pm z_2 \left(\sqrt{4 \cos^2 \theta - 4} \right)}{2}$$

- ⇒ $z_1 = z_2 (\cos \theta \pm i \sin \theta)$
- ⇒ $z_1 = z_2 e^{\pm i\theta}$
- ⇒ $z_1 = z_2 e^{i\theta}$ or $z_2 = z_1 e^{i\theta}$
- ⇒ z_1 is obtained by rotating z_2 anticlockwise through θ or z_2 is obtained by rotating z_1 anti-clockwise through θ .

In both the cases, $|z_1| = |z_2|$ and the angle between z_1 and z_2 is θ
Hence origin, z_1 and z_2 form an isosceles triangle with vertex at origin and vertical angle as θ

Example : 22

Find the locus of the point z which satisfies :

- (i) $2 < |z| \leq 3$
- (ii) $|z| = |z - i| = |z - 1|$
- (iii) $|z - 2| < |z - 6|$
- (iv) $\text{Arg} \left(\frac{z - 1 - i}{z - 2} \right) = \frac{\pi}{2}$

Solution

Important Note : $(z - z_0)$ represents an arrow going from a fixed point z_0 to a moving point z .

- (i) $2 < |z| \leq 3$
 $|z|$ is the length of vector from origin to the moving point z .
 $|z| > 2 \Rightarrow z$ is outside the circle $x^2 + y^2 = 4$
 $|z| \leq 3 \Rightarrow z$ is on or inside the circle $x^2 + y^2 = 9$
 ⇒ locus is the region between two circles as shown

- (ii) $|z - 0| = |z - i| = |z - 1|$
distance of moving point from origin
= distance from i
= distance from $1 + 0i$
 \Rightarrow the moving point is equidistant from vertices
 $z_1 = 0$, $z_2 = i$ and $z_3 = 1 + 0i$ of a triangle.
Hence it is at the circumcentre of this triangle
- (iii) $|z - 2| < |z - 6|$
 \Rightarrow distance of z from $z_1 = 2$ is less than its distance from $z_2 = 6$
 \Rightarrow z lies to the left of the right bisector of segment joining z_1 and z_2
- Alternatively : $|z + iy - 2| < |x + iy - 6|$
 $\Rightarrow \sqrt{(x-2)^2 + y^2} < \sqrt{(x-6)^2 + y^2}$
 $\Rightarrow (x-2)^2 - (x-6)^2 < 0$
 $\Rightarrow 2x - 8 < 0 \quad \Rightarrow \quad x < 4$
 $\Rightarrow \text{Re}(z) < 4$

Hence z lies in the region to the left of the line $x = 4$

- (iv) $\text{Arg} \left(\frac{z - z_1}{z - z_2} \right)$ is the angle between vectors joining the fixed points z_1 and z_2 to the moving point z .

$$\text{Arg} \left(\frac{z - z_1}{z - z_2} \right) = \pi/3 \quad z_1 = 1 + i, z_2 = 2$$

- \Rightarrow the point z moves such that the angle subtended at z by segment joining z_1 and z_2 is $\pi/3$
 \Rightarrow the locus is an arc of a circle. The equation of the locus can be found by taking $z = x + iy$.

$$\text{Arg} \left(\frac{x + iy - 1 - i}{x + iy - 2} \right) = \frac{\pi}{3}$$

$$\Rightarrow \tan^{-1} \left(\frac{y-1}{x-1} \right) - \tan^{-1} \left(\frac{y}{x-2} \right) = \frac{\pi}{3}$$

$$\Rightarrow \frac{\frac{y-1}{x-1} - \frac{y}{x-2}}{1 + \frac{(y-1)y}{(x-1)(x-2)}} = \sqrt{3}$$

$$\Rightarrow \frac{-x - y + 2}{x^2 - 3x + y^2 - y + 2} = \sqrt{3}$$

$$\Rightarrow \sqrt{3}(x^2 + y^2) - 3\sqrt{3} - 1)x - (\sqrt{3} - 1)y + 2\sqrt{3} - 2 = 0$$

Locus of z is the arc of this circle lying to the non-origin side of line joining $z_1 = 1 + i$ and $z_2 = 2$.

Example : 23

If $|z| \leq 1$, $|w| \leq 1$, show that : $|z - w|^2 \leq (|z| - |w|)^2 + (\text{Arg } z - \text{arg } w)^2$

Solution

Let O be the origin and points W and Z are represented by complex numbers z and w on the Argand plane.

Apply cosine rule in ΔOWZ i.e.

$$|w - z|^2 = |z|^2 + |w|^2 - 2|z||w|\cos\theta$$

$$= |z|^2 + |w|^2 - 2|z||w|\left(1 - 2\sin^2\frac{\theta}{2}\right)$$

$$= (|z| - |w|)^2 + 4|z||w|\sin^2\theta/2.$$

As $|z| \leq 1$ and $|w| \leq 1$, make RHS greater than LHS by replacing $|z| = 1$, $|w| = 1$

$$|w - z|^2 \leq (|z| - |w|)^2 + 4\sin^2\theta/2$$

On RHS, replace $\sin \theta/2$ ($\because \theta > \sin \theta$ for $\theta > 0$)

$$\begin{aligned} \Rightarrow |w - z|^2 &\leq (|z| - |w|)^2 + 4 \theta/2 \times \theta/2 \\ \Rightarrow |w - z|^2 &\leq (|z| - |w|)^2 + \theta^2 \\ \Rightarrow |w - z|^2 &\leq (|z| - |w|)^2 + (\text{Arg}(z) - \text{Arg}(w))^2 \end{aligned}$$

hence proved

Example : 24

If $iz^3 + z^2 - z + i = 0$, then show that $|z| = 1$.

Solution

Consider : $iz^3 + z^2 - z + i = 0$

By inspection, we can see that $z = i$ satisfies the above equation.

$\Rightarrow z - i$ is a factor of the LHS

Factoring LHS, we get : $(z - i)(iz^2 - 1) = 0$

$\Rightarrow z = i$ and $z^2 = 1/i = -i$

Case - 1

$$z = i \Rightarrow |z| = 1$$

Case - II

$$z^2 = -i$$

Take modulus of both sides,

$$|z|^2 = |-i| = 1 \Rightarrow |z| = 1$$

Hence, in both cases $|z| = 1$

Example : 25

If z_1 and z_2 are two complex numbers such that $\left| \frac{z_1 - z_2}{z_1 + z_2} \right| = 1$, Prove that $\frac{iz_1}{z_2} = k$, where k is a real number. Find the angle between the lines from the origin to the points $z_1 + z_2$ and $z_1 - z_2$ in terms of k .

Solution

$$\text{Consider } \left| \frac{z_1 - z_2}{z_1 + z_2} \right| = 1$$

Divide N and D on LHS by z_2 to get :

$$\Rightarrow \frac{\left| \frac{z_1 - 1}{z_2} \right|}{\left| \frac{z_1 + 1}{z_2} \right|} = 1 \Rightarrow \left| \frac{z_1 - 1}{z_2} \right| = \left| \frac{z_1 + 1}{z_2} \right|$$

$$\text{On squaring, } \left| \frac{z_1}{z_2} \right|^2 + 1 - 2 \text{Re} \left(\frac{z_1}{z_2} \right) = \left| \frac{z_1}{z_2} \right|^2 + 1 + 2 \text{Re} \left(\frac{z_1}{z_2} \right)$$

$$\Rightarrow 4 \text{Re} \left(\frac{z_1}{z_2} \right) = 0 \Rightarrow \frac{z_1}{z_2} \text{ is purely imaginary number.}$$

$$\Rightarrow \frac{z_1}{z_2} \text{ can be written as : } i \frac{z_1}{z_2} = k \text{ where } k \text{ is real number} \dots\dots\dots(i)$$

(ii) If θ is the angle between $z_1 - z_2$ and $z_1 + z_2$, then $\theta = \text{Arg} \frac{z_1 + z_2}{z_1 - z_2}$

$$\Rightarrow \theta = \text{Arg} \left[\frac{\frac{z_1}{z_2} + 1}{\frac{z_1}{z_2} - 1} \right]$$

Using (i), we get

$$\theta = \text{Arg} \left[\frac{-ik+1}{-ik-1} \right] = \text{Arg} \left[\frac{-1+ik}{1+ik} \right] = \text{Arg} \left[\frac{k^2-1+2ik}{1+k^2} \right]$$

$$\Rightarrow \theta = \tan^{-1} \frac{2k}{k^2-1}$$

Example : 26

For any $z_1, z_2 \in \mathbb{C}$, show that $|z_1 + z_2|^2 + |z_1 - z_2|^2 = 2|z_1|^2 + 2|z_2|^2$

Solution

Consider $\text{LHS} = |z_1 + z_2|^2 + |z_1 - z_2|^2$

$$\begin{aligned} \Rightarrow \text{LHS} &= (z_1 + z_2)(\overline{z_1 + z_2}) + (z_1 - z_2)(\overline{z_1 - z_2}) \\ &= (z_1 + z_2)(\bar{z}_1 + \bar{z}_2) + (z_1 - z_2)(\bar{z}_1 - \bar{z}_2) \\ &= (|z_1|^2 + |z_2|^2 + z_1 \bar{z}_2 + z_2 \bar{z}_1) + (|z_1|^2 + |z_2|^2 - z_1 \bar{z}_2 - z_2 \bar{z}_1) \\ &= 2|z_1|^2 + 2|z_2|^2 \end{aligned}$$

Example : 27

If $S_1 = {}^n C_0 + {}^n C_3 + {}^n C_6 + \dots$
 $S_2 = {}^n C_1 + {}^n C_2 + {}^n C_7 + \dots$
 $S_3 = {}^n C_4 + {}^n C_5 + {}^n C_8 + \dots$

each series being continued as far as possible, show that the values of S_1, S_2 and S_3 are $1/3(2^n + 2 \cos r\pi/3)$ where $r = n_1, n - 2, n + 2$ respectively and $n \in \mathbb{N}$.

Solution

Consider the identity :

$$(1+x)^n = C_0 + C_1 x + C_2 x^2 + C_3 x^3 + \dots + C_n x^n$$

Put $x = 1, x = \omega$ and $x = \omega^2$ in above identity to get :

$$2^n = C_0 + C_1 + C_2 + C_3 + \dots + C_n \quad \text{.....(i)}$$

$$(1+\omega)^n = C_0 + C_1 \omega + C_2 \omega^2 + C_3 \omega^3 + \dots + C_n \omega^n \quad \text{.....(ii)}$$

$$(1+\omega^2)^n = C_0 + C_1 \omega^2 + C_2 \omega + C_3 \omega^3 + \dots + C_n \omega^{2n} \quad \text{.....(iii)}$$

Find S_1

Add (i), (ii) and (iii) to get :

$$3C_0 + C_1(1+\omega+\omega^2) + C_2(1+\omega^2+\omega) + 3C_3 + \dots = 2^n + (1+\omega)^n + (1+\omega^2)^n$$

$$\Rightarrow 3C_0 + 3C_3 + 3C_6 + \dots = 2^n + \left(\frac{1+\sqrt{3}i}{2} \right)^n + \left(\frac{1+\sqrt{3}i}{2} \right)^n$$

$$\Rightarrow 3S_1 = 2^n + \left(\cos \frac{\pi}{3} + i \sin \frac{\pi}{3} \right)^n + \left(\cos \frac{\pi}{3} - i \sin \frac{\pi}{3} \right)^n$$

$$\Rightarrow S_1 = \frac{2^n + 2 \cos \frac{n\pi}{3}}{3} \quad \text{(using demoiivre's Law)}$$

Find S_2

Multiply (ii) with ω^2 , (iii) with ω and add to (i) to get :

$$C_0(1+\omega^2+\omega) + 3C_1 + C_2(1+\omega+\omega^2) + C_3(1+\omega^2+\omega) + \dots = 2^n + \omega^2(1+\omega)^n + \omega(1+\omega^2)^n$$

$$3C_1 + 3C_4 + 3C_7 + \dots = 2^n + \left(\cos \frac{2\pi}{3} + i \sin \frac{2\pi}{3} \right) \left(\cos \frac{n\pi}{3} - i \sin \frac{n\pi}{3} \right) + \left(\cos \frac{\pi}{3} + i \sin \frac{\pi}{3} \right) \left(\cos \frac{n\pi}{3} - i \sin \frac{n\pi}{3} \right)$$

$$\Rightarrow 3S_2 = 2^n + \cos \frac{(n-2)\pi}{3} + i \sin \frac{(n-2)\pi}{3} + \cos \frac{(n-2)\pi}{3} - i \sin \frac{(n-2)\pi}{3} = 2^n + 2 \cos \frac{(n-2)\pi}{3}$$

$$\Rightarrow S_2 = \frac{2^n + 2 \cos \frac{(n-2)\pi}{3}}{3}$$

Find S_3

Multiply (ii) by ω , (iii) with ω^2 and add to (i) to get

$$3(C_2 + C_5 + C_8 + \dots) = 2^n + 2 \cos \frac{(n+2)\pi}{3}$$

$$\Rightarrow S_3 = \frac{2^n + 2 \cos \frac{(n+2)\pi}{3}}{3}$$

Example : 28

Prove that the complex number z_1, z_2 and the origin form an isosceles triangle with vertical angle $2\pi/3$. If

$$z_1^2 + z_2^2 + z_1 z_2 = 0$$

Solution

Let A and B are the points represented by z_1 and z_2 respectively on the Argand plane

$$\text{Consider } z_1^2 + z_2^2 + z_1 z_2 = 0$$

On factoring LHS, we get :

$$(z_2 - \omega z_1)(z_2 - \omega^2 z_1) = 0$$

$$\Rightarrow z_2 = \omega z_1 \quad \text{or} \quad z_2 = \omega^2 z_1$$

consider $z_2 = \omega z_1$ (i)

Take modulus of both sides

$$|z_2| = |\omega z_1|$$

$$\Rightarrow |z_2| = |\omega| |z_1| = |z_1| \quad (\because |\omega| = 1)$$

$$\Rightarrow OA = OB \quad \Rightarrow \quad \Delta OAB \text{ is isosceles.}$$

Take argument on both sides,

$$\text{Arg}(z_2) = \text{Arg}(\omega z_1) = \text{Arg}(\omega) + \text{Arg}(z_1)$$

$$\Rightarrow \text{Arg}(z_2) - \text{Arg}(z_1) = 2\pi/3 \quad (\because \text{Arg}(\omega) = 2\pi/3)$$

$$\Rightarrow \angle AOB = 2\pi/3. \text{ Hence vertical angle} = \angle AOB = 2\pi/3.$$

Note : As $z_2 = \omega z_1 \Rightarrow z_2 = z_1 e^{i2\pi/3}$, we can directly conclude that z_2 is obtained by rotating z_1 through $2\pi/3$ in anti-clockwise direction

$$\Rightarrow \angle AOB = 2\pi/3 \quad \text{and} \quad OA = OB$$

Consider $z_2 = \omega^2 z_1$

Similarly show that ΔAOB is isosceles with vertical angle $2\pi/3$

Example : 29

For every real number $c \geq 0$, find all complex numbers z which satisfy the equation :

$$|z|^2 - 2iz + 2c(1+i) = 0.$$

Solution

Let $z = x + iy$

$$\Rightarrow (x^2 + y^2 + 2y + 2c) - i(2x - 2c) = 0$$

Comparing the real and imaginary parts, we get :

$$\Rightarrow x^2 + y^2 + 2y + 2c = 0 \quad \dots\dots\dots(i)$$

$$\text{and} \quad x = c \quad \dots\dots\dots(ii)$$

Solving (i) and (ii), we get

$$\Rightarrow y^2 + 2y + c^2 + 2c = 0$$

$$\Rightarrow y = \frac{-2 \pm \sqrt{4 - 4(c^2 + 2c)}}{2} = -1 \pm \sqrt{1 - c^2 - 2c}$$

as y is real, $1 - c^2 - 2c \geq 0$

$$\Rightarrow -\sqrt{2} - 1 \leq c \leq \sqrt{2} - 1$$

$$\Rightarrow c \leq \sqrt{2} - 1 \quad (\because c \geq 0)$$

\Rightarrow the solution is

$$z = x + iy = c + i \left(-1 \pm \sqrt{1 - c^2 - 2c} \right) \quad \text{for} \quad 0 \leq c \leq \sqrt{2} - 1$$

$$z = x + iy \equiv \text{no solution} \quad \text{for} \quad c > \sqrt{2} - 1$$

Example : 30

Let $\bar{b}z + b\bar{z} = c$, $b \neq 0$, be a line in the complex plane, where \bar{b} is the complex conjugate of b . If a point z_1 is the reflection of a point z_2 through the line, then show that $c = \bar{z}_1 b + z_2 \bar{b}$.

Solution

Since z_1 is image of z_2 in line $bz + \bar{b}\bar{z} = c$.
therefore mid-point of z_1 and z_2 should lie on the line i.e.

$$\frac{z_1 + z_2}{2} \text{ lies on } \bar{b}z + b\bar{z} = c$$

$$\Rightarrow \bar{b} \left(\frac{z_1 + z_2}{2} \right) + b \frac{\bar{z}_1 + \bar{z}_2}{2} = c$$

$$\Rightarrow \frac{\bar{b}z_1 + b\bar{z}_2}{2} + \frac{\bar{b}z_2 + b\bar{z}_1}{2} = c$$

Let z_b and z_c be two points on the given line.

$$\text{As } z_1 - z_2 \text{ is perpendicular to } z_b - z_c, \text{ we can take : } \frac{z_c - z_b}{|z_c - z_b|} e^{i\pi/2} = \frac{z_1 - z_2}{|z_1 - z_2|} \quad \dots\dots\dots(ii)$$

$$\Rightarrow \frac{z_1 - z_2}{z_c - z_b} = - \frac{\bar{z}_1 - \bar{z}_2}{\bar{z}_c - \bar{z}_b} \quad \Rightarrow \quad \frac{z_1 - z_2}{\bar{z}_1 - \bar{z}_2} = - \frac{z_c - z_b}{\bar{z}_c - \bar{z}_b}$$

As z_b and z_c also lie on line, we get :

$$\bar{b}z_b + b\bar{z}_b = c \quad \text{and} \quad \bar{b}z_c + b\bar{z}_c = c$$

$$\text{On subtracting, } \bar{b}(z_c - z_b) + b(\bar{z}_c - \bar{z}_b) = 0$$

$$\Rightarrow \frac{z_c - z_b}{\bar{z}_c - \bar{z}_b} = - \frac{b}{\bar{b}} \quad \dots\dots\dots(iii)$$

combining (ii) and (iii),

$$(z_1 - z_2) \bar{b} = b(\bar{z}_1 - \bar{z}_2)$$

$$\Rightarrow \bar{b}z_1 + b\bar{z}_2 = b\bar{z}_1 + \bar{b}z_2 \quad \dots\dots\dots(iv)$$

combining (i) and (iv) we get :

$$\frac{\bar{b}z_2 + b\bar{z}_1}{2} + \frac{\bar{b}z_2 + b\bar{z}_1}{2} = c$$

$$\Rightarrow \bar{b}z_2 + b\bar{z}_1 = c$$

Hence proved