

**Example : 1**

Express the following complex numbers in the trigonometric forms and hence calculate their principal arguments. Show the complex numbers on the Argand plane

(i)  $z_1 = -\sqrt{3} + i$       (ii)  $z_2 = -1 - \sqrt{3}i$       (iii)  $z_3 = 1 - i$

**Solution**

(i)  $z_1 = -\sqrt{3} + i$       ( $|z| = 2$ )

$$\Rightarrow z_1 = 2 \left( -\frac{\sqrt{2}}{3} + \frac{1}{2}i \right) \quad \left( \text{as } \cos \theta = -\frac{\sqrt{3}}{2}, \sin \theta = \frac{1}{2} \right)$$

$$\Rightarrow z_1 = 2 \left( \cos \frac{5\pi}{6} + i \sin \frac{5\pi}{6} \right) \quad \Rightarrow \quad \text{the argument} = \frac{5\pi}{6}$$

(ii)  $z_3 = -1 - \sqrt{3}i$       ( $|z| = 2$ )

$$\Rightarrow z_2 = 2 \left( -\frac{1}{2} - \frac{\sqrt{3}}{2}i \right) \quad \left( \cos \theta = -\frac{1}{2}, \sin \theta = -\frac{\sqrt{3}}{2} \right)$$

$$\Rightarrow z_2 = 2 \left[ \cos \left( \frac{-2\pi}{3} \right) + i \sin \left( \frac{-2\pi}{3} \right) \right]$$

$$\Rightarrow \text{argument} = \frac{-2\pi}{3}$$

(iii)  $z_3 = 1 - i$       ( $|z| = \sqrt{2}$ )

$$\Rightarrow z_3 = \sqrt{2} \left( \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i \right) \quad \left( \cos \theta = \frac{1}{\sqrt{2}}, \sin \theta = -\frac{1}{\sqrt{2}} \right)$$

$$\Rightarrow z_3 = \sqrt{2} \left[ \cos \left( \frac{-\pi}{4} \right) + i \sin \left( \frac{-\pi}{4} \right) \right]$$

$$\Rightarrow \text{argument} = \frac{-\pi}{4}$$

**Example : 2**

If  $z_1 = r_1 (\cos \alpha + i \sin \alpha)$  and  $z_2 = r_2 (\cos \beta + i \sin \beta)$ , show that :

(i)  $|z_1 z_2| = r_1 r_2$       (ii)  $\arg (z_1 z_2) = \alpha + \beta$

(iii)  $\left| \frac{z_1}{z_2} \right| = \frac{r_1}{r_2}$       (iv)  $\arg \left( \frac{z_1}{z_2} \right) = \alpha - \beta$

**Solution**

For (i) and (ii) :

$$\begin{aligned} z_1 z_2 &= r_1 r_2 (\cos \alpha + i \sin \alpha) (\cos \beta + i \sin \beta) \\ &= r_1 r_2 (\cos \alpha \cos \beta - \sin \alpha \sin \beta + i \sin \alpha \cos \beta + i \cos \alpha \sin \beta) \\ &= r_1 r_2 [\cos (\alpha + \beta) + i \sin (\alpha + \beta)] \end{aligned}$$

comparing with  $z = |z| (\cos \theta + i \sin \theta)$ , we get :

$$|z_1 z_2| = r_1 r_2 \quad \text{and} \quad \arg (z_1 z_2) = \alpha + \beta$$

For (iii) and (iv) :

$$\begin{aligned} \frac{z_1}{z_2} &= \frac{r_1 (\cos \alpha + i \sin \alpha)}{r_2 (\cos \beta + i \sin \beta)} \\ &= \frac{r_1}{r_2} (\cos \alpha + i \sin \alpha) (\cos \beta - i \sin \beta) \end{aligned}$$

$$= \frac{r_1}{r_2} [\cos \alpha \cos \beta + \sin \alpha \sin \beta + i \sin \alpha \cos \beta - i \cos \alpha \sin \beta]$$

$$= \frac{r_1}{r_2} [\cos (\alpha - \beta) + i \sin (\alpha + \beta)]$$

$$\Rightarrow \left| \frac{z_1}{z_2} \right| = \frac{r_1}{r_2} \quad \text{and} \quad \arg \left( \frac{z_1}{z_2} \right) = \alpha - \beta$$

### Example : 3

Show that  $|z - 2i| = 2\sqrt{2}$ , if  $\arg \left( \frac{z-2}{z+2} \right) = \frac{\pi}{4}$

#### Solution

Let  $z = x + yi$        $x, y \in \mathbb{R}$

$$\Rightarrow \arg \left( \frac{x-2+yi}{x+2+yi} \right) = \frac{\pi}{4}$$

$$\Rightarrow \arg \left[ \frac{(x-2+yi)(x+2-yi)}{(x+2)^2 + y^2} \right] = \frac{\pi}{4}$$

$$\Rightarrow \arg \left[ \frac{(x^2 - 4 + y^2) + 4yi}{(x+2)^2 + y^2} \right] = \frac{\pi}{4}$$

$$\Rightarrow \frac{4y}{x^2 - 4 + y^2} = \tan \frac{\pi}{4}$$

$$\Rightarrow x^2 + y^2 - 4y - 4 = 0$$

$$\Rightarrow x^2 + (y-2)^2 = 8$$

$$\Rightarrow |x + (y-2)i| = 2\sqrt{2}$$

$$\Rightarrow |z - 2i| = 2\sqrt{2}$$

### Example : 4

If  $\cos \alpha + \cos \beta + \cos \gamma = \sin \alpha + \sin \beta + \sin \gamma = 0$ , then show that :

- (i)  $\cos 3\alpha + \cos 3\beta + \cos 3\gamma = 3 \cos (\alpha + \beta + \gamma)$
- (ii)  $\sin 3\alpha + \sin 3\beta + \sin 3\gamma = 3 \sin (\alpha + \beta + \gamma)$
- (iii)  $\cos 2\alpha + \cos 2\beta + \cos 2\gamma = \sin 2\alpha + \sin 2\beta + \sin 2\gamma = 0$

#### Solution

For (i) and (ii) :

$$\text{Let } z_1 = \cos \alpha + i \sin \alpha \quad ;$$

$$z_2 = \cos \beta + i \sin \beta \quad ;$$

$$z_3 = \cos \gamma + i \sin \gamma$$

$$z_1 + z_2 + z_3 = \sum \cos \alpha + i \sum \sin \alpha = 0$$

for  $3\alpha, 3\beta, 3\gamma$  we have to consider  $z_1^3, z_2^3, z_3^3$

$$z_1^3 + z_2^3 + z_3^3 = (\cos \alpha + i \sin \alpha)^3 + (\cos \beta + i \sin \beta)^3 + (\cos \gamma + i \sin \gamma)^3$$

$$= (\cos 3\alpha + i \sin 3\alpha) + (\cos 3\beta + i \sin 3\beta) + (\cos 3\gamma + i \sin 3\gamma)$$

$$= (\cos 3\alpha + \cos 3\beta + \cos 3\gamma) + i (\sin 3\alpha + \sin 3\beta + \sin 3\gamma) \quad \dots\dots\dots(i)$$

Now  $z_1^3 + z_2^3 + z_3^3 = 3z_1 z_2 z_3$  because  $z_1 + z_2 + z_3 = 0$

$$\Rightarrow z_1^3 + z_2^3 + z_3^3 = 3 (\cos \alpha + i \sin \alpha) (\cos \beta + i \sin \beta) (\cos \gamma + i \sin \gamma)$$

$$z_1^3 + z_2^3 + z_3^3 = 3 [\cos (\alpha + \beta + \gamma) + i \sin (\alpha + \beta + \gamma)] \quad \dots\dots\dots(ii)$$

Equating the RHS of (i) and (ii), we get :

$$\sum \cos 3\alpha + i \sum \sin 3\alpha = 3 \cos (\alpha + \beta + \gamma) + 3 i \sin (\alpha + \beta + \gamma)$$

Equating real and imaginary parts,

$$\sum \cos 3\alpha = 3 \cos (\alpha + \beta + \gamma) \quad \text{and} \quad \sum \sin 3\alpha = 3 \sin (\alpha + \beta + \gamma)$$

For (iii) :

Consider  $z_1^2 + z_2^2 + z_3^2 = (z_1 + z_2 + z_3)^2 - 2(z_1 z_2 + z_2 z_3 + z_3 z_1)$

$$= 0 - 2z_1 z_2 z_3 \left( \frac{1}{z_1} + \frac{1}{z_2} + \frac{1}{z_3} \right)$$

$$= 2z_1 z_2 z_3 \left[ \frac{1}{\cos \alpha + i \sin \alpha} + \frac{1}{\cos \beta + i \sin \beta} + \frac{1}{\cos \gamma + i \sin \gamma} \right]$$

$$= -2z_1 z_2 z_3 [\cos \alpha - i \sin \alpha + \cos \beta - i \sin \beta + \cos \gamma - i \sin \gamma]$$

$$= -2z_1 z_2 z_3 [\sum \cos \alpha - i \sum \sin \alpha]$$

$$= -2z_1 z_2 z_3 [0 - i(0)] = 0$$

$$\Rightarrow (\cos \alpha + i \sin \alpha)^2 + (\cos \beta + i \sin \beta)^2 + (\cos \gamma + i \sin \gamma)^2 = 0$$

$$\Rightarrow (\cos \alpha + i \sin 2\alpha) + (\cos 2\beta + i \sin 2\beta) + (\cos 2\gamma + i \sin 2\gamma) = 0$$

$$\Rightarrow \sum \cos 2\alpha = 0 \quad \text{and} \quad \sum \sin 2\alpha = 0$$

**Example : 5**

Express  $\sin 5\theta$  in terms of  $\sin \theta$  and hence show that  $\sin 36^\circ$  is a root of the equation  $16x^4 + 20x^2 + 5 = 0$ .

**Solution**

Expand  $(\cos \theta + i \sin \theta)^5$  using binomial theorem.

$$(\cos \theta + i \sin \theta)^5 = {}^5C_0 \cos^5 \theta + 5 {}^5C_1 \cos^4 \theta (i \sin \theta) + \dots + 5 {}^5C_4 i^4 \cos \theta \sin^4 \theta + {}^5C_5 i^5 \sin^5 \theta$$

using DeMoivre's theorem on L.H.S. :

$$(\cos 5\theta + i \sin 5\theta) = (\cos^5 \theta - 10 \cos^3 \theta \sin^2 \theta + 5 \cos \theta \sin^4 \theta) + i [5 \cos^4 \theta \sin \theta - 10 \cos^2 \theta \sin^3 \theta + \sin^5 \theta]$$

Equating imaginary parts :

$$\sin 5\theta = \sin \theta [5 \cos^4 \theta - 10 \cos^2 \theta \sin^2 \theta + \sin^4 \theta]$$

$$\sin 5\theta = \sin \theta [5(1 + \sin^4 \theta - 2 \sin^2 \theta) - 10(1 - \sin^2 \theta) \sin^2 \theta] + \sin^4 \theta$$

$$\sin 5\theta = 16 \sin^5 \theta - 20 \sin^3 \theta + 5 \sin \theta$$

$$\text{for } \theta = 36^\circ, \quad \sin 5\theta = \sin 180^\circ = 0$$

$$\Rightarrow 16 \sin^5 36^\circ - 20 \sin^3 36^\circ + 5 \sin 36^\circ = 0$$

$$\Rightarrow \sin 36^\circ \text{ is a root of } 16x^5 - 20x^3 + 5x = 0$$

$$\text{i.e. } 16x^4 - 20x^2 + 5 = 0$$

**Example : 6**

If  $(1 + x)^n = P_0 + P_1 x + P_2 x^2 + \dots + P_n x^n$ , then show that

(a)  $P_0 - P_2 + P_4 - \dots = 2^{n/2} \cos(n\pi/4)$

(b)  $P_1 - P_3 + P_5 - \dots = 2^{n/2} \sin(n\pi/4)$

**Solution**

Consider the identity

$$(1 + x)^n = P_0 + P_1 x + P_2 x^2 + P_3 x^3 + \dots + P_n x^n$$

Put  $x = i$  on both the sides

$$(1 + i)^n = P_0 + P_1 i + P_2 i^2 + P_3 i^3 + \dots + P_n i^n$$

$$\left[ \sqrt{2} \left( \cos \frac{\pi}{4} + i \sin \frac{\pi}{4} \right) \right]^n = (P_0 - P_2 + P_4 + \dots) + i (P_1 - P_3 + P_5 + \dots)$$

$$2^{n/2} \left( \cos \frac{n\pi}{4} + i \sin \frac{n\pi}{4} \right) = (P_0 - P_2 + P_4 + \dots) + i (P_1 - P_3 + P_5 + \dots)$$

equate the real and imaginary parts.

$$P_0 - P_2 + P_4 - P_6 + \dots = 2^{n/2} \cos \frac{n\pi}{4}$$

$$P_1 - P_3 + P_5 - \dots = 2^{n/2} \sin \frac{n\pi}{4}$$

**Example : 7**

If a, b, c and d are the roots of the equation  $x^4 + P_1x^3 + P_2x^2 + P_3x + P_4 = 0$ , then show that :  
 $(1 + a^2) (1 + b^2) (1 + c^2) (1 + d^2) = (1 - P_2 + P_4)^2 + (P_3 - P_1)^2$

**Solution**

As a, b, c and d are the roots of the given equation :

$\Rightarrow (x - a), (x - b), (x - c)$  and  $(x - d)$  are the factors of LHS

$\Rightarrow x^4 + P_1x^3 + P_2x^2 + P_3x + P_4 = (x - a) (x - b) (x - c) (x - d)$  is an identity .....(i)

Put  $x = i$  on both sides :

$$i^4 + P_1i^3 + P_2i^2 + P_3i + P_4 = (i - a) (i - b) (i - c) (i - d)$$

$$(1 - P_2 + P_4) + i (P_3 - P_1) = (i - a) (i - b) (i - c) (i - d) \quad \text{.....(ii)}$$

Put  $x = -i$  in (i) :

$$i^4 - P_1i^3 + P_2i^2 - P_3i + P_4 = (-i - a) (i - b) (-i - c) (-i - d)$$

$$(1 - P_2 + P_4) - i (P_3 - P_1) = (-i - a) (-i - b) (-i - c) (-i - d) \quad \text{.....(iii)}$$

multiply (ii) and (iii) to get

$$(1 - P_2 + P_4)^2 + (P_3 - P_1)^2 = (1 + a^2) (1 + b^2) (1 + c^2) (1 + d^2)$$

**Example : 8**

Show that  $|z_1 \pm z_2|^2 = |z_1|^2 + |z_2|^2 \pm 2 \operatorname{Re} (z_1 \bar{z}_2)$ .

**Solution**

$$|z_1 \pm z_2|^2 = (z_1 \pm z_2) (\bar{z}_1 \pm \bar{z}_2)$$

$$= z_1 \bar{z}_2 + z_2 \bar{z}_1 \pm (z_1 \bar{z}_2 + \bar{z}_1 z_2)$$

$$= |z_1|^2 + |z_2|^2 \pm (z_1 \bar{z}_2 + \bar{z}_1 z_2)$$

$$= |z_1|^2 + |z_2|^2 \pm 2 \operatorname{Re} (z_1 \bar{z}_2) \quad \text{because } z + \bar{z} = 2 \operatorname{Re} (z)$$

**Example : 9**

If 1,  $\omega$ ,  $\omega^2$  are cube roots of unity. Show that :

$$(1 - \omega + \omega^2) (1 - \omega^2 + \omega^4) (1 - \omega^4 + \omega^8) \dots \dots \dots 2n \text{ factors} = 2^{2n}$$

**Solution**

$$\text{LHS} = (1 - \omega + \omega^2) (1 - \omega^2 + \omega^4) (1 - \omega^4 + \omega^8) \dots \dots \dots 2n \text{ factors}$$

using  $\omega^4 = \omega^{16} = \dots = \omega$  and  $\omega^8 = \omega^{32} = \dots = \omega^2$

$$\text{L.H.S.} = (1 - \omega + \omega^2) (1 - \omega^2 + \omega) (1 - \omega + \omega^2) (1 - \omega^2 + \omega) \dots \dots \dots 2n \text{ factors.}$$

$$\text{L.H.S.} = [(1 - \omega + \omega^2) (1 - \omega^2 + \omega)]^n = [(-2\omega) (-2\omega^2)]^n$$

$$\text{L.H.S.} = 2^{2n} = \text{R.H.S.}$$

**Example : 10**

Prove that the area of the triangle whose vertices are the points  $z_1, z_2, z_3$  on the argand diagram is :

$$\sum \left[ \frac{(z_2 - z_3) |z_1|^2}{4iz_1} \right]$$

**Solution**

Let the vertices of the triangle be

$$A (x_1, y_1) \quad : \quad z_1 = x_1 + iy_1$$

$$B (x_2, y_2) \quad : \quad z_2 = x_2 + iy_2$$

$$C (x_3, y_3) \quad : \quad z_3 = x_3 + iy_3$$

Area of triangle ABC is :

$$\Delta = \frac{1}{2} \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

We have to express the area in terms of  $z_1, z_2$  and  $z_3$ .

Operating  $C_1 \rightarrow C_1 + iC_2$  (properties of Determinants)

$$\Delta = \frac{1}{2} \begin{vmatrix} x_1 + iy_1 & y_1 & 1 \\ x_2 + iy_2 & y_2 & 1 \\ x_3 + iy_3 & y_3 & 1 \end{vmatrix}$$

$$\Delta = \frac{1}{2} \begin{vmatrix} z_1 & y_1 & 1 \\ z_2 & y_2 & 1 \\ z_3 & y_3 & 1 \end{vmatrix}$$

$$\Delta = \frac{1}{4i} \begin{vmatrix} z_1 & z_1 - \bar{z}_1 & 1 \\ z_2 & z_2 - \bar{z}_2 & 1 \\ z_3 & z_3 - \bar{z}_3 & 1 \end{vmatrix}$$

Operating  $C_2 \rightarrow C_2 - C_1$  (properties of Determinants)

$$\Delta = \frac{1}{4i} \begin{vmatrix} z_1 & \bar{z}_1 & 1 \\ z_2 & \bar{z}_2 & 1 \\ z_3 & z_3 & 1 \end{vmatrix}$$

$$\Rightarrow \frac{1}{4i} [\bar{z}_1 (z_2 - z_3) + \bar{z}_2 (z_1 - z_3) - \bar{z}_3 (z_1 - z_2)]$$

$$\Rightarrow \Delta = \frac{1}{4i} [\bar{z}_1 (z_2 - z_3) + \bar{z}_2 (z_3 - z_1) - \bar{z}_3 (z_1 - z_2)]$$

$$\Rightarrow \Delta = \frac{1}{4i} \sum \bar{z}_1 (z_2 - z_3)$$

$$\Rightarrow \Delta = \frac{1}{4i} \sum \left[ \frac{|\bar{z}_1|^2 (z_2 - z_3)}{z_1} \right]$$

### Example : 11

Show that the sum of nth roots of unity is zero.

#### Solution

Let  $S = 1 + e^{i2\pi/n} + e^{i4\pi/n} + \dots + e^{i2\pi(n-1)/n}$   
the series on the RHS is a GP

$$\Rightarrow S = \frac{1 \left( 1 - e^{i\frac{2\pi}{n}n} \right)}{1 - e^{i\frac{2\pi}{n}}} \Rightarrow S = \frac{1 - e^{i2\pi}}{1 - e^{i\frac{2\pi}{n}}}$$

$$\Rightarrow S = \frac{1 - 1}{1 - e^{i\frac{2\pi}{n}}} = 0$$

### Example : 12

Find the value of :  $\sum_{r=1}^{r=6} \left[ \sin \frac{2\pi r}{7} - i \cos \frac{2\pi r}{7} \right]$

#### Solution

$$\text{Let } S = \sum_{r=1}^{r=6} \left[ \sin \frac{2\pi r}{7} - i \cos \frac{2\pi r}{7} \right] = -i \sum_{r=1}^{r=6} \left[ \cos \frac{2\pi r}{7} + i \sin \frac{2\pi r}{7} \right]$$

$$\begin{aligned}
&= -i \sum_{r=1}^{r=6} e^{i\frac{2\pi r}{7}} = -i \left[ \sum_{r=0}^{r=6} e^{i\frac{2\pi r}{7}} - 1 \right] \\
&= -i (\text{sum of 7th roots of unity} - 1) \\
&= -i(0 - 1) = i
\end{aligned}$$

**Example : 13**

Find the sixth roots of  $z = i$

**Solution**

$$z = 1 \left( \cos \frac{\pi}{2} + i \sin \frac{\pi}{2} \right)$$

$$z^{1/6} = 1^{1/6} \left( \cos \frac{\pi/2 + 2k\pi}{6} + i \sin \frac{\pi/2 + 2k\pi}{6} \right) \quad \text{where } k = 0, 1, 2, 3, 4, 5$$

⇒ The sixth roots are :

$$k = 0 \quad \Rightarrow \quad z_n = \left( \frac{\pi}{12} + i \sin \frac{\pi}{12} \right)$$

$$k = 1 \quad \Rightarrow \quad z_1 = \cos \frac{5\pi}{12} + i \sin \frac{5\pi}{12}$$

$$k = 2 \quad \Rightarrow \quad z_2 = \cos \frac{9\pi}{12} + i \sin \frac{9\pi}{12}$$

$$k = 3 \quad \Rightarrow \quad z_3 = \cos \frac{13\pi}{12} + i \sin \frac{13\pi}{12} = \cos \frac{11\pi}{12} - i \sin \frac{11\pi}{12}$$

$$k = 4 \quad \Rightarrow \quad z_4 = \cos \frac{17\pi}{12} + i \sin \frac{17\pi}{12} = -\cos \frac{5\pi}{12} + i \sin \frac{5\pi}{12}$$

$$k = 5 \quad \Rightarrow \quad z_5 = \cos \frac{21\pi}{12} + i \sin \frac{21\pi}{12} = \cos \frac{3\pi}{12} - i \sin \frac{3\pi}{12}$$

**Example : 14**

Prove that  $(x + y)^n - x^n - y^n$  is divisible by  $xy(x + y)(x^2 + y^2 + xy)$  if  $n$  is odd but not a multiple of 3.

**Solution**

Let  $f(x) = (x + y)^n - x^n - y^n$

$f(0) = (0 + y)^n - (0)^n - y^n = 0$

⇒  $(x - 0)$  is a factor of  $f(x)$

⇒  $x$  is a factor of  $f(x)$

By symmetry  $y$  is also a factor of  $f(x)$

$f(-y) = (-y + y)^n - (-y)^n - y^n = 0$  (because  $n$  is odd)

⇒  $(x + y)$  is also a factor of  $f(x)$ .

Now consider  $f(\omega y)$

$f(\omega y) = (\omega y + y)^n - (\omega y)^n - y^n$

$= y^n (-\omega^2)^n - \omega^n y^n - y^n$

$= y^n [-\omega^{2n} - \omega^n - 1]$  (because  $n$  is odd)

$= -y^n [\omega^{2n} + \omega^n + 1]$

$n$  is not a multiple of 3.

⇒  $n = 3k + 1$  or  $n = 3k + 2$  where  $k$  is an integer

⇒  $[\omega^{2n} + \omega^n + 1] = 0$  (for both cases)

⇒  $f(\omega y) = 0$

⇒  $(x - \omega y)$  is also a factor of  $f(x)$

Similarly we can show that  $f(\omega^2 y) = 0$

⇒  $(x - \omega^2 y)$  is also a factor of  $f(x)$

Combining all the factors :

we get :  $xy(x+y)(x-\omega^2y)(x-\omega^2y)$  is a factor of  $f(x)$   
 now  $(x-\omega y)(x-\omega^2y) = x^2 + xy + y^2$   
 $\Rightarrow f(x)$  is divisible by  $xy(x+y)(x-\omega y)(x-\omega^2y)$

### Example : 15

Interpret the following equations geometrically on the Argand plane :

(i)  $|z - 2 - 3i| = 4$                       (ii)  $|z - 1| + |z + 1| = 4$

(iii)  $\arg\left(\frac{z-1}{z+1}\right) = \frac{\pi}{4}$                       (iv)  $\frac{\pi}{6} < \arg(z) < \frac{\pi}{3}$

### Solution

To interpret the equations geometrically, we will convert them to Cartesian form in terms of  $x$  and  $y$  coordinates by substituting  $z = x + iy$

(i)  $|x + iy - 2 - 3i| = 4$   
 $\Rightarrow (x-2)^2 + (y-3)^2 = 4^2$   
 $\Rightarrow$  the equation represents a circle centred at  $(2, 3)$  of radius 4 units

(ii)  $|x + iy - 1| = |x + iy + 1| = 4$   
 $\Rightarrow \sqrt{(x-1)^2 + y^2} + \sqrt{(x+1)^2 + y^2} = 4$

simplify to get :  $\frac{x^2}{4} + \frac{y^2}{3} = 1$

$\Rightarrow$  the equation represents an ellipse centred at  $(0, 0)$

(iii)  $\text{Arg}\left(\frac{x+iy-1}{x+iy+1}\right) = \frac{\pi}{4}$

$\Rightarrow \text{Arg}(x+iy-1) - \text{Arg}(x+iy+1) = \frac{\pi}{4}$

$\Rightarrow \frac{\frac{y}{x-1} - \frac{y}{x+1}}{1 + \frac{y^2}{x^2-1}} = \tan \frac{\pi}{4} \Rightarrow \frac{2y}{x^2+y^2-1} = 1$

$\Rightarrow x^2 + y^2 - 2y - 1 = 0$

$\Rightarrow$  the equation represents a circle centred at  $z = 0 + i$  and of radius  $= \sqrt{2}$ .

(iv)  $\frac{\pi}{6} < \tan^{-1}\left(\frac{y}{x}\right) < \frac{\pi}{3}$

$\Rightarrow \frac{1}{\sqrt{3}} x < y < \sqrt{3} x$

$\Rightarrow$  this inequation represents the region between the lines :  
 $y = \sqrt{3} x$  and  $y = (1/\sqrt{3}) x$  in  $Q_1$

### Example : 16

Find the complex number having least positive argument and satisfying  $|z - 5i| \leq 3$

### Solution

We will analyse the problem geometrically.

All complex numbers ( $z$ ) satisfying  $|z - 5i| \leq 3$  lies on or inside the circle of radius 3 centred at  $z_0 = 5i$ .

The complex number having least positive argument in this region is at the point of contact of a tangent drawn from origin to the circle.

From triangle OAC

$OA = \sqrt{5^2 - 3^2} = 4$

$$\text{and } \theta_{\min} = \sin^{-1} \left( \frac{OA}{OC} \right) = \sin^{-1} \left( \frac{4}{5} \right)$$

the complex number at A has modulus 4 and argument  $\sin^{-1} 4/5$

$$\Rightarrow z_A = 4 (\cos \theta + i \sin \theta) = 4 \left( \frac{3}{5} + i \frac{4}{5} \right)$$

$$\Rightarrow z_A = \frac{12}{5} + i \frac{16}{5}$$

### Example : 17

Show that the area of the triangle on the Argand plane formed by the complex numbers  $z$ ,  $iz$  and  $(z + iz)$  is  $(1/2) |z|^2$ .

#### Solution

$$iz = ze^{i\pi/2}$$

$\Rightarrow iz$  is the vector obtained by rotating  $z$  in anti-clockwise direction through  $90^\circ$

As  $|iz| = |i| |z|$ , the triangle is an isosceles right angled triangle.

$$\text{Area} = 1/2 = \text{base} \times \text{height} = 1/2 |z| |iz|$$

### Example : 18

If  $|z|^2 = 5$ , find the area of the triangle formed by the complex numbers  $z$ ,  $\omega z$  and  $z + \omega z$  as its sides.

#### Solution

$$\omega z = ze^{i2\pi/3} \quad \text{and} \quad |\omega z| = |z|$$

$\Rightarrow \omega z$  is the vector obtained by rotating vector  $z$  anti-clockwise through an angle of  $120^\circ$

As seen from the figure, the triangle formed is equilateral because angle between equal sides is  $60^\circ$

$$\Rightarrow \text{Area} = \sqrt{3}/4 (\text{side})^2 = \sqrt{3}/4 |z|^2 = \sqrt{3} \text{ sq. units.}$$

Note that the third side is

$$z + \omega z = (1 + \omega) z = -\omega^2 z = e^{i\pi} e^{-i2\pi/3} z = z e^{i\pi/3}$$

$\Rightarrow$  this vector is obtained by rotating the vector  $z$  anticlockwise through  $60^\circ$ . This can be verified from the figure

### Example : 19

Show that  $z_1, z_2, z_3$  represent the vertices of an equilateral triangle if and only if :

$$z_1^2 + z_2^2 + z_3^2 - z_1 z_2 - z_2 z_3 - z_3 z_1 = 0$$

#### Solution

The problem has two parts :

- (i) If the triangle is equilateral then prove the condition
- (ii) If the condition is given then prove the triangle is equilateral.

#### Part (i)

If the triangle ABC is equilateral, the vector BC can be obtained by rotating AB anti-clockwise through  $120^\circ$

$$\Rightarrow (z_3 - z_2) = (z_2 - z_1) e^{i2\pi/3}$$

$$\Rightarrow z_3 - z_2 = (z_2 - z_1) \omega$$

$$\Rightarrow z_1 \omega - z_2 \omega - z_2 + z_3 = 0$$

$$\Rightarrow z_1 - z_2 \omega^3 - z_2 \omega^2 + z_3 \omega^2 = 0$$

$$\Rightarrow z_1 - (1 + \omega^2) z_2 + \omega^2 z_3 = 0$$

$$\Rightarrow z_1 + \omega z_2 + \omega^2 z_3 = 0$$

Taking LHS :

$$z_1^2 + z_2^2 + z_3^2 - z_1 z_2 - z_2 z_3 - z_3 z_1 = (z_1 + \omega z_2 + \omega^2 z_3) (z_1 + \omega^3 z_2 + \omega z_3) = 0 \quad (\text{using the above proved result})$$

#### Part (ii)

Give that :

$$z_1^2 + z_2^2 + z_3^2 - z_1 z_2 - z_2 z_3 - z_3 z_1 = 0$$

$$\Rightarrow (z_1 + \omega z_2 + \omega^2 z_3) (z_1 + \omega^3 z_2 + \omega z_3) = 0$$

$$\Rightarrow (z_1 + \omega z_2 + \omega^2 z_3 = 0 \quad \text{OR} \quad (z_1 + \omega^3 z_2 + \omega z_3) = 0$$

#### Case (1) :

$$(z_1 + \omega z_2 + \omega^2 z_3) = 0$$

$$\Rightarrow z_1 + \omega z_2 + (-1 - \omega) z_3 = 0$$

- ⇒  $(z_1 - z_3) = \omega (z_3 - z_2)$
- ⇒  $(z_1 - z_2)$  is obtained by rotating the vector  $(z_3 - z_2)$  anti-clockwise through  $120^\circ$
- ⇒  $|z_1 - z_3| = |z_3 - z_2|$  and the angle inside the triangle is  $60^\circ$
- ⇒ triangle ABC is equilateral

Case (2) :

- $(z_1 + \omega^2 z_2 + \omega z_3) = 0$
- ⇒  $z_1 + \omega z_3 + (-1 - \omega) z_2 = 0$
- ⇒  $(z_1 - z_2) = \omega (z_2 - z_3)$
- ⇒  $|z_1 - z_2|$  is obtained by rotating the vector  $(z_3 - z_2)$  anti-clockwise through  $120^\circ$
- ⇒  $|z_1 - z_2| = |z_2 - z_3|$  and the angle inside the triangle is  $60^\circ$
- ⇒ triangle ABC is equilateral

**Example : 20**

Let the complex numbers  $z_1, z_2$  and  $z_3$  be the vertices of an equilateral triangle. Let  $z_0$  be the circumcentre of the triangle. Prove that :  $z_1^2 + z_2^2 + z_3^2 = 3z_0^2$ .

**Solution**

For an equilateral triangle with vertices  $z_1, z_2$  and  $z_3$  :

$$z_1^2 + z_2^2 + z_3^2 - z_1 z_2 - z_2 z_3 - z_3 z_1 = 0 \quad \dots\dots\dots(i)$$

As circumcentre coincides with centroid,  $z_0$  is centroid also.

- ⇒  $z_0 = (z_1 + z_2 + z_3)/3$
- ⇒  $9z_0^2 = z_1^2 + z_2^2 + z_3^2 + 2(z_1 z_2 + z_2 z_3 + z_3 z_1)$

using (i), we have

- ⇒  $9z_0^2 = z_1^2 + z_2^2 + z_3^2 + 2(z_1^2 + z_2^2 + z_3^2)$
- ⇒  $9z_0^2 = 3(z_1^2 + z_2^2 + z_3^2)$
- ⇒  $3z_0^2 = z_1^2 + z_2^2 + z_3^2$

**Example : 21**

If  $z_1^2 + z_2^2 - 2z_1 z_2 \cos \theta = 0$ , then the origin,  $z_1, z_2$  from vertices of an isosceles triangle with vertical angle  $\theta$ .

**Solution**

- $z_1^2 + z_2^2 - 2z_1 z_2 \cos \theta = 0$
- ⇒  $z_1^2 - (2z_2 \cos \theta) z_1 + z_2^2 = 0$

Solving as a quadratic in  $z_1$ , we get :

$$z_1 = \frac{2z_2 \cos \theta \pm z_2 \left( \sqrt{4 \cos^2 \theta - 4} \right)}{2}$$

- ⇒  $z_1 = z_2 (\cos \theta \pm i \sin \theta)$
- ⇒  $z_1 = z_2 e^{\pm i\theta}$
- ⇒  $z_1 = z_2 e^{i\theta}$  or  $z_2 = z_1 e^{i\theta}$
- ⇒  $z_1$  is obtained by rotating  $z_2$  anticlockwise through  $\theta$  or  $z_2$  is obtained by rotating  $z_1$  anti-clockwise through  $\theta$ .

In both the cases,  $|z_1| = |z_2|$  and the angle between  $z_1$  and  $z_2$  is  $\theta$   
Hence origin,  $z_1$  and  $z_2$  form an isosceles triangle with vertex at origin and vertical angle as  $\theta$

**Example : 22**

Find the locus of the point  $z$  which satisfies :

- (i)  $2 < |z| \leq 3$
- (ii)  $|z| = |z - i| = |z - 1|$
- (iii)  $|z - 2| < |z - 6|$
- (iv)  $\text{Arg} \left( \frac{z - 1 - i}{z - 2} \right) = \frac{\pi}{2}$

**Solution**

Important Note :  $(z - z_0)$  represents an arrow going from a fixed point  $z_0$  to a moving point  $z$ .

- (i)  $2 < |z| \leq 3$   
  - $|z|$  is the length of vector from origin to the moving point  $z$ .
  - $|z| > 2$  ⇒  $z$  is outside the circle  $x^2 + y^2 = 4$
  - $|z| \leq 3$  ⇒  $z$  is on or inside the circle  $x^2 + y^2 = 9$
- ⇒ locus is the region between two circles as shown

- (ii)  $|z - 0| = |z - i| = |z - 1|$   
distance of moving point from origin  
= distance from  $i$   
= distance from  $1 + 0i$   
 $\Rightarrow$  the moving point is equidistant from vertices  
 $z_1 = 0$ ,  $z_2 = i$  and  $z_3 = 1 + 0i$  of a triangle.  
Hence it is at the circumcentre of this triangle
- (iii)  $|z - 2| < |z - 6|$   
 $\Rightarrow$  distance of  $z$  from  $z_1 = 2$  is less than its distance from  $z_2 = 6$   
 $\Rightarrow$   $z$  lies to the left of the right bisector of segment joining  $z_1$  and  $z_2$
- Alternatively :  $|z + iy - 2| < |x + iy - 6|$   
 $\Rightarrow \sqrt{(x-2)^2 + y^2} < \sqrt{(x-6)^2 + y^2}$   
 $\Rightarrow (x-2)^2 - (x-6)^2 < 0$   
 $\Rightarrow 2x - 8 < 0 \quad \Rightarrow \quad x < 4$   
 $\Rightarrow \text{Re}(z) < 4$

Hence  $z$  lies in the region to the left of the line  $x = 4$

- (iv)  $\text{Arg} \left( \frac{z - z_1}{z - z_2} \right)$  is the angle between vectors joining the fixed points  $z_1$  and  $z_2$  to the moving point  $z$ .

$$\text{Arg} \left( \frac{z - z_1}{z - z_2} \right) = \pi/3 \quad z_1 = 1 + i, z_2 = 2$$

- $\Rightarrow$  the point  $z$  moves such that the angle subtended at  $z$  by segment joining  $z_1$  and  $z_2$  is  $\pi/3$   
 $\Rightarrow$  the locus is an arc of a circle. The equation of the locus can be found by taking  $z = x + iy$ .

$$\text{Arg} \left( \frac{x + iy - 1 - i}{x + iy - 2} \right) = \frac{\pi}{3}$$

$$\Rightarrow \tan^{-1} \left( \frac{y-1}{x-1} \right) - \tan^{-1} \left( \frac{y}{x-2} \right) = \frac{\pi}{3}$$

$$\Rightarrow \frac{\frac{y-1}{x-1} - \frac{y}{x-2}}{1 + \frac{(y-1)y}{(x-1)(x-2)}} = \sqrt{3}$$

$$\Rightarrow \frac{-x - y + 2}{x^2 - 3x + y^2 - y + 2} = \sqrt{3}$$

$$\Rightarrow \sqrt{3}(x^2 + y^2) - 3\sqrt{3} - 1)x - (\sqrt{3} - 1)y + 2\sqrt{3} - 2 = 0$$

Locus of  $z$  is the arc of this circle lying to the non-origin side of line joining  $z_1 = 1 + i$  and  $z_2 = 2$ .

### Example : 23

If  $|z| \leq 1$ ,  $|w| \leq 1$ , show that :  $|z - w|^2 \leq (|z| - |w|)^2 + (\text{Arg } z - \text{arg } w)^2$

### Solution

Let  $O$  be the origin and points  $W$  and  $Z$  are represented by complex numbers  $z$  and  $w$  on the Argand plane.

Apply cosine rule in  $\Delta OWZ$  i.e.

$$|w - z|^2 = |z|^2 + |w|^2 - 2|z||w|\cos\theta$$

$$= |z|^2 + |w|^2 - 2|z||w|\left(1 - 2\sin^2\frac{\theta}{2}\right)$$

$$= (|z| - |w|)^2 + 4|z||w|\sin^2\theta/2.$$

As  $|z| \leq 1$  and  $|w| \leq 1$ , make RHS greater than LHS by replacing  $|z| = 1$ ,  $|w| = 1$

$$|w - z|^2 \leq (|z| - |w|)^2 + 4\sin^2\theta/2$$

On RHS, replace  $\sin \theta/2$  ( $\because \theta > \sin \theta$  for  $\theta > 0$ )

$$\begin{aligned} \Rightarrow |w - z|^2 &\leq (|z| - |w|)^2 + 4 \theta/2 \times \theta/2 \\ \Rightarrow |w - z|^2 &\leq (|z| - |w|)^2 + \theta^2 \\ \Rightarrow |w - z|^2 &\leq (|z| - |w|)^2 + (\text{Arg}(z) - \text{Arg}(w))^2 \end{aligned}$$

hence proved

**Example : 24**

If  $iz^3 + z^2 - z + i = 0$ , then show that  $|z| = 1$ .

**Solution**

Consider :  $iz^3 + z^2 - z + i = 0$

By inspection, we can see that  $z = i$  satisfies the above equation.

$\Rightarrow z - i$  is a factor of the LHS

Factoring LHS, we get :  $(z - i)(iz^2 - 1) = 0$

$\Rightarrow z = i$  and  $z^2 = 1/i = -i$

**Case - 1**

$$z = i \Rightarrow |z| = 1$$

**Case - II**

$$z^2 = -i$$

Take modulus of both sides,

$$|z|^2 = |-i| = 1 \Rightarrow |z| = 1$$

Hence, in both cases  $|z| = 1$

**Example : 25**

If  $z_1$  and  $z_2$  are two complex numbers such that  $\left| \frac{z_1 - z_2}{z_1 + z_2} \right| = 1$ , Prove that  $\frac{iz_1}{z_2} = k$ , where  $k$  is a real number. Find the angle between the lines from the origin to the points  $z_1 + z_2$  and  $z_1 - z_2$  in terms of  $k$ .

**Solution**

$$\text{Consider } \left| \frac{z_1 - z_2}{z_1 + z_2} \right| = 1$$

Divide N and D on LHS by  $z_2$  to get :

$$\Rightarrow \frac{\left| \frac{z_1 - 1}{z_2} \right|}{\left| \frac{z_1 + 1}{z_2} \right|} = 1 \Rightarrow \left| \frac{z_1 - 1}{z_2} \right| = \left| \frac{z_1 + 1}{z_2} \right|$$

$$\text{On squaring, } \left| \frac{z_1}{z_2} \right|^2 + 1 - 2 \text{Re} \left( \frac{z_1}{z_2} \right) = \left| \frac{z_1}{z_2} \right|^2 + 1 + 2 \text{Re} \left( \frac{z_1}{z_2} \right)$$

$$\Rightarrow 4 \text{Re} \left( \frac{z_1}{z_2} \right) = 0 \Rightarrow \frac{z_1}{z_2} \text{ is purely imaginary number.}$$

$$\Rightarrow \frac{z_1}{z_2} \text{ can be written as : } i \frac{z_1}{z_2} = k \text{ where } k \text{ is real number} \dots\dots\dots(i)$$

(ii) If  $\theta$  is the angle between  $z_1 - z_2$  and  $z_1 + z_2$ , then  $\theta = \text{Arg} \frac{z_1 + z_2}{z_1 - z_2}$

$$\Rightarrow \theta = \text{Arg} \left[ \frac{\frac{z_1}{z_2} + 1}{\frac{z_1}{z_2} - 1} \right]$$

Using (i), we get

$$\theta = \text{Arg} \left[ \frac{-ik+1}{-ik-1} \right] = \text{Arg} \left[ \frac{-1+ik}{1+ik} \right] = \text{Arg} \left[ \frac{k^2-1+2ik}{1+k^2} \right]$$

$$\Rightarrow \theta = \tan^{-1} \frac{2k}{k^2-1}$$

**Example : 26**

For any  $z_1, z_2 \in \mathbb{C}$ , show that  $|z_1 + z_2|^2 + |z_1 - z_2|^2 = 2|z_1|^2 + 2|z_2|^2$

**Solution**

Consider  $\text{LHS} = |z_1 + z_2|^2 + |z_1 - z_2|^2$

$$\begin{aligned} \Rightarrow \text{LHS} &= (z_1 + z_2)(\overline{z_1 + z_2}) + (z_1 - z_2)(\overline{z_1 - z_2}) \\ &= (z_1 + z_2)(\bar{z}_1 + \bar{z}_2) + (z_1 - z_2)(\bar{z}_1 - \bar{z}_2) \\ &= (|z_1|^2 + |z_2|^2 + z_1 \bar{z}_2 + z_2 \bar{z}_1) + (|z_1|^2 + |z_2|^2 - z_1 \bar{z}_2 - z_2 \bar{z}_1) \\ &= 2|z_1|^2 + 2|z_2|^2 \end{aligned}$$

**Example : 27**

If  $S_1 = {}^nC_0 + {}^nC_3 + {}^nC_6 + \dots$   
 $S_2 = {}^nC_1 + {}^nC_2 + {}^nC_7 + \dots$   
 $S_3 = {}^nC_2 + {}^nC_5 + {}^nC_8 + \dots$

each series being continued as far as possible, show that the values of  $S_1, S_2$  and  $S_3$  are  $1/3(2^n + 2 \cos r\pi/3)$  where  $r = n_1, n - 2, n + 2$  respectively and  $n \in \mathbb{N}$ .

**Solution**

Consider the identity :

$$(1+x)^n = C_0 + C_1 x + C_2 x^2 + C_3 x^3 + \dots + C_n x^n$$

Put  $x = 1, x = \omega$  and  $x = \omega^2$  in above identity to get :

$$\begin{aligned} 2^n &= C_0 + C_1 + C_2 + C_3 + \dots + C_n \quad \dots\dots\dots \text{(i)} \\ (1+\omega)^n &= C_0 + C_1 \omega + C_2 \omega^2 + C_3 \omega^3 + \dots + C_n \omega^n \quad \dots\dots\dots \text{(ii)} \\ (1+\omega^2)^n &= C_0 + C_1 \omega^2 + C_2 \omega + C_3 + \dots + C_n \omega^{2n} \quad \dots\dots\dots \text{(iii)} \end{aligned}$$

Find  $S_1$

Add (i), (ii) and (iii) to get :

$$3C_0 + C_1(1 + \omega + \omega^2) + C_2(1 + \omega^2 + \omega) + 3C_3 + \dots = 2^n + (1 + \omega)^n + (1 + \omega^2)^n$$

$$\Rightarrow 3C_0 + 3C_3 + 3C_6 + \dots = 2^n + \left( \frac{1 + \sqrt{3}i}{2} \right)^n + \left( \frac{1 + \sqrt{3}i}{2} \right)^n$$

$$\Rightarrow 3S_1 = 2^n + \left( \cos \frac{\pi}{3} + i \sin \frac{\pi}{3} \right)^n + \left( \cos \frac{\pi}{3} - i \sin \frac{\pi}{3} \right)^n$$

$$\Rightarrow S_1 = \frac{2^n + 2 \cos \frac{n\pi}{3}}{3} \quad \text{(using demoiivre's Law)}$$

Find  $S_2$

Multiply (ii) with  $\omega^2$ , (iii) with  $\omega$  and add to (i) to get :

$$C_0(1 + \omega^2 + \omega) + 3C_1 + C_2(1 + \omega + \omega^2) + C_3(1 + \omega^2 + \omega) + \dots = 2^n + \omega^2(1 + \omega)^n + \omega(1 + \omega^2)^n$$

$$3C_1 + 3C_4 + 3C_7 + \dots = 2^n + \left( \cos \frac{2\pi}{3} + i \sin \frac{2\pi}{3} \right) \left( \cos \frac{n\pi}{3} - i \sin \frac{n\pi}{3} \right) + \left( \cos \frac{\pi}{3} + i \sin \frac{\pi}{3} \right) \left( \cos \frac{n\pi}{3} - i \sin \frac{n\pi}{3} \right)$$

$$\Rightarrow 3S_2 = 2^n + \cos \frac{(n-2)\pi}{3} + i \sin \frac{(n-2)\pi}{3} + \cos \frac{(n-2)\pi}{3} - i \sin \frac{(n-2)\pi}{3} = 2^n + 2 \cos \frac{(n-2)\pi}{3}$$

$$\Rightarrow S_2 = \frac{2^n + 2 \cos \frac{(n-2)\pi}{3}}{3}$$

Find  $S_3$

Multiply (ii) by  $\omega$ , (iii) with  $\omega^2$  and add to (i) to get

$$3(C_2 + C_5 + C_8 + \dots) = 2^n + 2 \cos \frac{(n+2)\pi}{3}$$

$$\Rightarrow S_3 = \frac{2^n + 2 \cos \frac{(n+2)\pi}{3}}{3}$$

### Example : 28

Prove that the complex number  $z_1, z_2$  and the origin form an isosceles triangle with vertical angle  $2\pi/3$ . If

$$z_1^2 + z_2^2 + z_1 z_2 = 0$$

### Solution

Let A and B are the points represented by  $z_1$  and  $z_2$  respectively on the Argand plane

$$\text{Consider } z_1^2 + z_2^2 + z_1 z_2 = 0$$

On factoring LHS, we get :

$$(z_2 - \omega z_1)(z_2 - \omega^2 z_1) = 0$$

$$\Rightarrow z_2 = \omega z_1 \quad \text{or} \quad z_2 = \omega^2 z_1$$

consider  $z_2 = \omega z_1$  .....(i)

Take modulus of both sides

$$\Rightarrow |z_2| = |\omega z_1|$$

$$\Rightarrow |z_2| = |\omega| |z_1| = |z_1| \quad (\because |\omega| = 1)$$

$$\Rightarrow OA = OB \quad \Rightarrow \quad \Delta OAB \text{ is isosceles.}$$

Take argument on both sides,

$$\text{Arg}(z_2) = \text{Arg}(\omega z_1) = \text{Arg}(\omega) + \text{Arg}(z_1)$$

$$\Rightarrow \text{Arg}(z_2) - \text{Arg}(z_1) = 2\pi/3 \quad (\because \text{Arg}(\omega) = 2\pi/3)$$

$$\Rightarrow \angle AOB = 2\pi/3. \text{ Hence vertical angle} = \angle AOB = 2\pi/3.$$

Note : As  $z_2 = \omega z_1 \Rightarrow z_2 = z_1 e^{i2\pi/3}$ , we can directly conclude that  $z_2$  is obtained by rotating  $z_1$  through  $2\pi/3$  in anti-clockwise direction

$$\Rightarrow \angle AOB = 2\pi/3 \quad \text{and} \quad OA = OB$$

Consider  $z_2 = \omega^2 z_1$

Similarly show that  $\Delta AOB$  is isosceles with vertical angle  $2\pi/3$

### Example : 29

For every real number  $c \geq 0$ , find all complex numbers  $z$  which satisfy the equation :

$$|z|^2 - 2iz + 2c(1+i) = 0.$$

### Solution

Let  $z = x + iy$

$$\Rightarrow (x^2 + y^2 + 2y + 2c) - i(2x - 2c) = 0$$

Comparing the real and imaginary parts, we get :

$$\Rightarrow x^2 + y^2 + 2y + 2c = 0 \quad \dots\dots\dots(i)$$

$$\text{and} \quad x = c \quad \dots\dots\dots(ii)$$

Solving (i) and (ii), we get

$$\Rightarrow y^2 + 2y + c^2 + 2c = 0$$

$$\Rightarrow y = \frac{-2 \pm \sqrt{4 - 4(c^2 + 2c)}}{2} = -1 \pm \sqrt{1 - c^2 - 2c}$$

as  $y$  is real,  $1 - c^2 - 2c \geq 0$

$$\Rightarrow -\sqrt{2} - 1 \leq c \leq \sqrt{2} - 1$$

$$\Rightarrow c \leq \sqrt{2} - 1 \quad (\because c \geq 0)$$

$\Rightarrow$  the solution is

$$z = x + iy = c + i \left( -1 \pm \sqrt{1 - c^2 - 2c} \right) \quad \text{for} \quad 0 \leq c \leq \sqrt{2} - 1$$

$$z = x + iy \equiv \text{no solution} \quad \text{for} \quad c > \sqrt{2} - 1$$

**Example : 30**

Let  $\bar{b}z + b\bar{z} = c$ ,  $b \neq 0$ , be a line in the complex plane, where  $\bar{b}$  is the complex conjugate of  $b$ . If a point  $z_1$  is the reflection of a point  $z_2$  through the line, then show that  $c = \bar{z}_1 b + z_2 \bar{b}$ .

**Solution**

Since  $z_1$  is image of  $z_2$  in line  $bz + \bar{b}\bar{z} = c$ .  
therefore mid-point of  $z_1$  and  $z_2$  should lie on the line i.e.

$$\frac{z_1 + z_2}{2} \text{ lies on } \bar{b}z + b\bar{z} = c$$

$$\Rightarrow \bar{b} \left( \frac{z_1 + z_2}{2} \right) + b \frac{\bar{z}_1 + \bar{z}_2}{2} = c$$

$$\Rightarrow \frac{\bar{b}z_1 + b\bar{z}_2}{2} + \frac{\bar{b}z_2 + b\bar{z}_1}{2} = c$$

Let  $z_b$  and  $z_c$  be two points on the given line.

$$\text{As } z_1 - z_2 \text{ is perpendicular to } z_b - z_c, \text{ we can take : } \frac{z_c - z_b}{|z_c - z_b|} e^{i\pi/2} = \frac{z_1 - z_2}{|z_1 - z_2|} \quad \dots\dots\dots(ii)$$

$$\Rightarrow \frac{z_1 - z_2}{z_c - z_b} = - \frac{\bar{z}_1 - \bar{z}_2}{\bar{z}_c - \bar{z}_b} \quad \Rightarrow \quad \frac{z_1 - z_2}{\bar{z}_1 - \bar{z}_2} = - \frac{z_c - z_b}{\bar{z}_c - \bar{z}_b}$$

As  $z_b$  and  $z_c$  also lie on line, we get :

$$\bar{b}z_b + b\bar{z}_b = c \quad \text{and} \quad \bar{b}z_c + b\bar{z}_c = c$$

$$\text{On subtracting, } \bar{b}(z_c - z_b) + b(\bar{z}_c - \bar{z}_b) = 0$$

$$\Rightarrow \frac{z_c - z_b}{\bar{z}_c - \bar{z}_b} = - \frac{b}{\bar{b}} \quad \dots\dots\dots(iii)$$

combining (ii) and (iii),

$$(z_1 - z_2) \bar{b} = b(\bar{z}_1 - \bar{z}_2)$$

$$\Rightarrow \bar{b}z_1 + b\bar{z}_2 = b\bar{z}_1 + \bar{b}z_2 \quad \dots\dots\dots(iv)$$

combining (i) and (iv) we get :

$$\frac{\bar{b}z_2 + b\bar{z}_1}{2} + \frac{\bar{b}z_2 + b\bar{z}_1}{2} = c$$

$$\Rightarrow \bar{b}z_2 + b\bar{z}_1 = c$$

Hence proved