Example : 1

What does the equation $x^2 - 5xy + 4y^2 = 0$ represent?

Solution

$x^2 - 5xy + 4y^2 = 0$

$\Rightarrow$ $x^2 - 4xy - xy + 4y^2 = 0$

$\Rightarrow$ $(x - 4y)(x - y) = 0$

$\Rightarrow$ the equation represent two straight lines through origin whose equation are $x - 4y = 0$ and $x - y = 0$

Example : 2

Find the area formed by the triangle whose sides are $y^2 - 9xy + 18x^2 = 0$ and $y = 9$

Solution

$y^2 - 9xy + 18x^2 = 0$

$\Rightarrow$ $(y - 3x)(y - 6x) = 0$

$\Rightarrow$ the sides of the triangle are $y - 3x = 0$ and $y - 6x = 0$ and $y - 9 = 0$

$\Rightarrow$ By solving these simultaneously, we get the vertices as $A \equiv (0, 0)$ $B \equiv (3/2, 9)$ $C \equiv (3, 9)$

$$\text{Area} = \frac{1}{2}\begin{vmatrix} 0 & 0 & 1 \\ \frac{3}{2} & 9 & 1 \\ 3 & 9 & 1 \end{vmatrix} = \frac{27}{4} \text{ sq. units.}$$

Example : 3

Find the angle between the lines $x^2 + 4y^2 - 7xy = 0$

Solution

Using the result given in section 1.3, we get :

$$\text{Angle between the lines} = \theta = \tan^{-1}\frac{2\sqrt{h^2 - ab}}{a + b} = \tan^{-1}\left[\frac{2\sqrt{\left(\frac{-7}{2}\right)^2 - 1(4)}}{1 + 4}\right] \quad \tan^{-1}\left[\frac{\sqrt{33}}{5}\right]$$

Example : 4

Find the equation of pair of lines through origin which form an equilateral triangle with the lines $Ax + By + C = 0$. Also find the area of this equilateral triangle.

Solution

Let PQ be the side of the equilateral triangle lying on the line $Ax + By + C = 0$

Let m be the slope of line through origin and making an angle of 60º with $Ax + By + C = 0$

$\Rightarrow$ m is the slopes of OP or OQ

$\Rightarrow$ As the triangle is equilateral, $Ax + By + C = 0$ line makes an angle of 60º with OP and OQ

i.e. $$\tan 60^\circ = \left|\frac{m(-A/B)}{1 + m\left(\frac{-A}{B}\right)}\right| \quad \Rightarrow \quad 3 = \left(\frac{mB + A}{B - mA}\right)^2 \quad \text{..........(i)}$$

This quadratic will give two values of m which are slopes of OP and OQ.

As OP and OQ pass through origin, their equations can be taken as : $y = mx$ ....(ii)

Since we have to find the equation of OP and OQ, we will not find values of m but we will eliminate m between (i) and (ii) to directly get the equation of the pair of lines : OP and OQ

$$\Rightarrow \quad 3 = \left(\frac{By/x + A}{B - yA/x}\right)^2 \quad \Rightarrow \quad 3 = \left(\frac{By + Ax}{Bx - yA}\right)^2$$

$\Rightarrow$ $3(B^2x^2 + y^2A^2 - 2ABxy) = (B^2y^2 + A^2x^2 + 2ABxy)$

$\Rightarrow$ $(A^2 - 3B^2)\,x^2 + 8ABxy + (B^2 - 3A^2)\,y^2 = 0$ is the pair of lines through origin makes an equilateral triangle (OPQ) with $Ax + By + C = 0$

Area of equilateral $\Delta OPQ = \frac{\sqrt{3}}{4}\,(\text{side})^2 = \frac{\sqrt{3}}{4}\left(\frac{P}{\sin 60}\right)^2$ where P = altitude.

$\Rightarrow$ $\text{area} = \frac{\sqrt{3}}{4} \times \frac{4}{3}\,P^2 = \frac{1}{\sqrt{3}}\,P^2 = \frac{1}{\sqrt{3}}\left[\frac{|C|}{\sqrt{A^2 + B^2}}\right]^2 = \frac{C^2}{\sqrt{3}(A^2 + B^2)}$

Example : 5

If a pair of lines $x^2 - 2pxy - y^2 = 0$ and $x^2 - 2qxy - y^2 = 0$ is such that each pair bisects the angle between the other pair, prove that $pq = -1$

Solution

The pair of bisectors for $x^2 - 2pxy - y^2 = 0$ is : $\frac{x^2 - y^2}{1 - (-1)} = \frac{xy}{-p}$

$\Rightarrow$ $x^2 - y^2 = \frac{2xy}{-p}$

$\Rightarrow$ $x^2 + \frac{2}{p}\,xy - y^2 = 0$

As $x^2 + \frac{2}{p}\,xy - y^2 = 0$ and $x^2 - 2qxy - y^2 = 0$ coincide, we have

$\frac{1}{1} = \frac{2/p}{-2q} = \frac{-1}{-1}$

$\Rightarrow$ $\frac{2}{p} = -2q$ $\Rightarrow$ $pq = -1$

Example : 6

Prove that the angle between one of the lines given by $ax^2 + 2hxy + by^2 = 0$ and one of the lines $ax^2 + 2hxy + by^2 + \lambda(x^2 + y^2) = 0$ is equal to the angle between the other two lines of the system.

Solution

Let $L_1\,L_2$ be one pair and P_1P_2 be the other pair.
If the angle between L_1P_1 is equal to the angle between L_2P_2,
the pair of bisectors of L_1L_2 is same as that of P_1P_2

$\Rightarrow$ Pair of bisectors of P_1P_2 is $\frac{x^2 - y^2}{(a + \lambda) - (b + \lambda)} = \frac{xy}{h}$

$\Rightarrow$ $\frac{x^2 - y^2}{x - b} = \frac{xy}{h}$

Which is same as the bisector pair of L_1L_2
Hence the statement is proved.

Example : 7

Show that the orthocentre of the triangle formed by the lines $ax^2 + 2hxy + by^2 = 0$ and $\ell x + my = 1$ is given by $\frac{x}{\ell} = \frac{y}{m} = \frac{a + b}{am^2 - 2h\ell m + b\ell^2}$.

Solution

Let the triangle be OBC where O is origin and BC is the line $\ell x + my = 1$.
$\Rightarrow$ The equation of pair of lines OB and OC is $ax^2 + 2hxy + by^2 = 0$.

The equation of the altitude from O to BC is :

$$y - 0 = m/\ell\ (x - 0)$$

$\Rightarrow$ $mx - \ell y = 0$ ....(i)

Let equation of OB be $y - m_1x = 0$ and that of OC be $y - m_2x = 0$

$\Rightarrow$ $B \equiv \left[\frac{1}{\ell + mm_1}, \frac{m_1}{\ell + mm_1}\right]$

Slope of altitude from B to OC is $-1/m_2$

$\Rightarrow$ equation of altitude from B is :

$$y - \frac{m_1}{\ell + mm_1} = \frac{-1}{m_2}\left[x - \frac{1}{\ell + mm_1}\right]$$

$\Rightarrow$ $(l + mm_1)\ x + m_2\ (\ell + mm_1)\ y - (1 + m_1m_2) = 0$ ....(ii)

Solving (i) and (ii), we get orthocentre

$$\frac{x}{-\ell(1 + m_1m_2)} = \frac{y}{-m(1 + m_1m_2)} = \frac{1}{-\ell(\ell + mm_1) - m(\ell + mm_1)m_2}$$

using values of m_1m_2 and $m_1 + m_2$, we get :

$\Rightarrow$ $\frac{x}{\ell} = \frac{y}{m} = \frac{-(1 + a/b)}{-\ell^2 - m^2m_1m_2 - \ell m(m_1 + m_2)} = \frac{a + b}{b\ell^2 + am^2 - 2h\ell m}$

Example : 8

Prove that the equation $6x^2 - xy - 12y^2 - 8x + 29y - 14 = 0$ represent a pair of lines. Find the equations of each line.

Solution

Using the result given in section 2.1, we get

$$\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = \begin{vmatrix} 6 & -1/2 & -4 \\ -1/2 & -6 & \frac{29}{2} \\ -4 & \frac{29}{2} & -14 \end{vmatrix} = 0$$

Hence the given equation represents a pair of lines.

To find the equation of each line, we have to factories the LHS. We first factories the second degree term.

The second degree terms in the expression are :

$$6x^2 - xy - 12y^2 = 6x^2 - 9xy + 8xy - 12y^2 = (3x + 4y)\ (2x - 3y).$$

Let the two factors be $3x + 4y + C_1$ and $2x - 3y + C_2$.

$\Rightarrow$ $6x^2 - xy - 12y^2 - 8x + 29y - 14 = (3x + 4y + C_1)\ (2x - 3y + C_2)$

Comparing the coefficients of x and y, we get :

$-8 = 3C_2 + 2C_1$ and $29 = 4C_2 - 3C_1$

Solving for C_1 and C_2 , we get :

$C_2 = 2$ and $C_1 = -7$

$\Rightarrow$ the lines are $3x + 4y - 7 = 0$ and $2x - 3y + 2 = 0$

Example : 9

Find the equation of the lines joining the origin to the points of intersection of the line $4x - 3y = 10$ with the circle $x^2 + y^2 + 3x - 6y - 20 = 0$ and show that they are perpendicular.

Solution

To find equation of pair of lines joining origin to the points of intersection of given circle and line, we will make the equation of circle homogeneous by using : $1 = \frac{4x - 3y}{10}$

$\Rightarrow$ the pair of lines is : $x^2 + y^2 + (3x - 6y)\left(\frac{4x - 3y}{10}\right) - 20\left(\frac{4x - 3y}{10}\right)^2 = 0$

$\Rightarrow$ $10x^2 + 15xy - 10y^2 = 0$

Coefficient x^2 + coefficient of $y^2 = 10 - 10 = 0$

$\Rightarrow$ The lines of the pair are perpendicular.

This question can also be asked as :

["Show that the chord $4x - 3y = 10$ of the circle $x^2 + y^2\ 3x - 6y - 20 = 0$ subtends a right angle at origin."]

Example : 10

A variable chord of the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ always subtends a right angle at origin. Find the locus of the foot of the perpendicular drawn from origin to this chord.

Solution

Let the variable chord be $\ell x + my = 1$ where ℓ, m are changing quantities (i.e. parameters that change with the moving chord)

Let $P(x_1, y_1)$ be the foot of the perpendicular from origin to the chord.

If AB is the chord, then the equation of pair OA and OB is :

$x^2 + y^2 + (2gx + 2fy)(\ell x + my) + c(\ell x + my)^2 = 0$

$\Rightarrow$ $x^2(1 + 2g\ell + c\ell^2) + y^2(1 + 2fm + cm^2) + (2gm + 2f\ell) + 2c\ell m)\,xy = 0$

As OA is perpendicular OB,

coefficient of x^2 + coefficient of $y^2 = 0$

$\Rightarrow$ $(1 + 2g\ell + c\ell^2) + (1 + 2fm + cm^2) = 0$

As P lies on AB, $\ell x_1 + my_1 = 1$

As OP $\perp$ AB $\left(\frac{y_1}{x_1}\right)\left(\frac{-\ell}{m}\right) = -1$

We have to eliminate ℓ, m using (i), (ii) and (iii)

From (ii) and (iii), we get $m = \frac{y_1}{x_1^2 + y_1^2}$ and $\ell = \frac{x_1}{x_1^2 + y_1^2}$

Now from (i), we get :

$$1 + \frac{2gx_1}{x_1^2 + y_1^2} + \frac{cx_1^2}{(x_1^2 + y_1^2)^2} + \frac{2fy_1}{x_1^2 + y_1^2} + 1 + \frac{cy_1^2}{(x_1^2 + y_1^2)^2} = 0$$

$\Rightarrow$ $2(x_1^2 + y_1^2) + 2gx_1 + 2fy_1 + c = 0$

$\Rightarrow$ the locus of P is : $2(x^2 + y^2) + 2gx + 2fy + c = 0$

Example : 11

Show that the locus of a point, such that two of the three normals drawn from it to the parabola $y^2 = 4ax$ are perpendicular is $y^2 = a(x - 3a)$.

Solution

Let $P \equiv (x_1, y_1)$ be the point from where normals AP, BP, CP are drawn to $y^2 = 4ax$.

Let $y = mx - 2am - 2m^3$ be one of these normals

P lies on it $\Rightarrow$ $y_1 = mx_1 - am - am^3$.

Slopes m_1, m_2, m_3 of AP, BP, CP are roots of the cubic

$y_1 = mx_1 - 2am - am^2$

$\Rightarrow$ $am^3 + (2a - x_1)\,m + y_1 = 0$

$\Rightarrow$ $m_1 + m_2 + m_3 = 0$

$\Rightarrow$ $m_1m_2 + m_2m_3 + m_3m_1 = \frac{2a - x_1}{a}$

$\Rightarrow$ $m_1m_2m_3 = -\frac{y_1}{a}$

As two of the three normals are perpendicular, we take $m_1m_2 = -1$ (i.e. we assume AP perpendicular BP)

To get the locus, we have to eliminate m_1, m_2, m_3 .

$m_1m_2 + m_2m_3 + m_3m_1 = \dfrac{2a - x_1}{a}$

$\Rightarrow \quad -1 + m_3(-m_3) = \dfrac{2a - x_1}{a}$

$\Rightarrow \quad -1 - \left(\dfrac{+y_1}{a}\right)^2 = \dfrac{2a - x_1}{a}$ [using $m_1m_2m_3 = -y_1/a$ and $m_1m_2 = -1$]

$\Rightarrow \quad a^2 + y_1^2 = -2a^2 + ax_1$

$\Rightarrow \quad y_1^2 = a(x_1 - 3a)$

$\Rightarrow \quad y^2 = a(x - 3a)$ is the required locus.

Example : 12

Suppose that the normals drawn at three different points on the parabola $y^2 = 4x$ pass through the point (h, k). Show that $h > 2$

Solution

Let the normal(s) be $y = mx - 2am - 2m^3$. they pass through (h, k).

$\Rightarrow \quad k = mh - 2am - am^3$.

The three roots m_1, m_2, m_3 of this cubic are the slope of the three normals. Taking a = 1, we get :

$m^3 + (2 - h)m + k = 0$

$\Rightarrow \quad m_1 + m_2 + m_3 = 0$

$\Rightarrow \quad m_1m_2 + m_2m_3 \;\; m_3m_1 = 2 - h$

$\Rightarrow \quad m_1m_2m_3 = -k$

As m_1, m_2, m_3 are real, $m_1^2 + m_2^2 + m_3^2 > 0$ (and not all are zero)

$\Rightarrow \quad (m_1 + m_2 + m_3)^2 - 2(m_1m_2 + m_2m_3 + m_3m_1) > 0$

$\Rightarrow \quad 0 - 2(2 - h) > 0$

$\Rightarrow \quad h > 2$.

Example : 13

If the normals to the parabola $y^2 = 4ax$ at three points P, Q and R meet at A and S be the focus, prove that SP . SQ . SR = $a(SA)^2$.

Solution

Since the slopes of normals are not involved but the coordinates of P, Q, R are important, we take the normal as :

$tx + y = 2at = at^3$

Let A ≡ (h, k)

$\Rightarrow \quad t_1, t_2, t_3$ are roots of the $th + k = 2at^3$ i.e. $at^3 + (2a - h)t - k = 0$

$\Rightarrow \quad t_1 + t_2 + t_3 = 0$

$\Rightarrow \quad t_1t_2 + t_2t_3 + t_3t_1 = \dfrac{2a - h}{a}$

$\Rightarrow \quad t_1t_2t_3 = k/a$

Remainder that distance of point P(t) from focus and from directrix is $SP = a(1 + t^2)$

$\Rightarrow \quad SP = a(1 + t_1^2)$, $SQ = a(1 + t^2)$, $SR = a(1 + t_3)^2$

SP, SQ, SR = $a^3 (t_1^2 + t_2^2 + t_3^2) + (t_1^2 t_2^2 + t_2^2 t_3^2 + t_3^2 t_1^2) + (t_1^2 + t_2^2 + t_3^2) + 1$

we can see that : $t_1^2 + t_2^2 + t_3^2 = (t_1 + t_2 + t_3)^2 - 2\Sigma t_1t_2 = 0 - 2\dfrac{(2a - h)}{a}$

and also $\Sigma t_1^2 t_2^2 = (\Sigma t_1t_2)^2 - 2\Sigma (t_1t_2)(t_2t_3)$ [using : $\Sigma a^2 = (\Sigma a)^2 - 2\Sigma ab$]

$= \dfrac{(2a - h)^2}{a^2} - 2t_1t_2t_3(0) = \dfrac{(2a - h)^2}{a^2}$

$\Rightarrow \quad$ SP, SQ, SR = $a^3 \left\{\dfrac{k^2}{a^2} + \dfrac{(2a - h)^2}{a^2} + \dfrac{2h - 4a}{a} + 1\right\} = a\{(h - a)^2 + k^2\} = aSA^2$

Example : 14

Show that the tangent and the normal at a point P on the parabola $y^2 = 4ax$ are the bisectors of the angle between the focal radius SP and the perpendicular from P on the directrix.

Solution

Let $P \equiv (at^2, 2at)$, $S \equiv (a, 0)$

Equation of SP is : $y - 0 = \frac{2at - 0}{at^2 - a}(x - a)$

$\Rightarrow \quad 2tx + (1 - t^2)y + (-2at) = 0$ ....(i)

Equation of PM is : $y - 2sat = 0$ ....(ii)

Angle bisectors of (i) and (ii) are :

$$\frac{y - 2at}{\sqrt{0+1}} = \pm \frac{2tx + (1-t^2)y - 2at}{\sqrt{4t^2 + (1-t^2)^2}}$$

$\Rightarrow \quad y - 2at = \pm \frac{2tx + (1-t^2)y - 2at}{1+t^2}$

$\Rightarrow \quad ty = x + at^2$ and $tx + y = 2at + at^3$

$\Rightarrow$ tangent and normal at P are bisectors of SP and PM.

Alternate Method :

Let the tangent at P meet X-axis in Q.

As MP is parallel to X-axis, $\angle MPQ = \angle PQS$

Now we can find SP and SQ.

$SP = \sqrt{(1 - at^2)^2 + (0 - 2at)^2} = a(1 + t^2)$

Equation of PQ is $ty = x + at^2$

$\Rightarrow \quad Q \equiv (-at^2, 0)$

$\Rightarrow \quad SQ = \sqrt{(a + at^2) + 0} = a(1 + t^2)$

$\Rightarrow \quad SP = SQ$

$\Rightarrow \quad \angle SPQ = \angle SQP = \angle MPQ$

Hence PQ bisects $\angle SPM$

It obviously follows that normal bisects exterior angle.

Example : 15

In the parabola $y^2 = 4ax$, the tangent at the point P, whose abscissa is equal to the latus rectum meets the axis in T and the normal at P cuts the parabola again in Q. Prove that PT : PQ = 4 : 5

Solution

Latus rectum = $x_p = 4a$

Let $P \equiv (at^2, 2at)$

$\Rightarrow \quad at^2 = 4a \quad \Rightarrow \quad t = \pm 2$

We can do the problem by taking only one of the values.

Let t = 2

$\Rightarrow \quad P \equiv (4a, 4a)$

$\Rightarrow$ tangent at P is $2y = x + 4a$

T lies on X-axis, $\Rightarrow \quad T \equiv (-4a, 0)$

$\Rightarrow \quad PT = \sqrt{(8a)^2 + (4a)^2} = 4a\sqrt{5}$

Let us nor find PQ.

If normal at P(t) cuts parabola again at $Q(t_1)$, then $t_1 = -t - 2/t$

$\Rightarrow \quad t_1 = -2 - 2/2 = -3$

$\Rightarrow \quad Q \equiv (9a, -6a)$

$\Rightarrow \quad PQ = \sqrt{25a^2 + 100a^2} = 5a\sqrt{5}$

$\Rightarrow \quad PT : PQ = 4 : 5$

Example : 16

A variable chord PQ of $y^2 = 4ax$ subtends a right angle at vertex. Prove that the locus of the point of intersection of normals at P, Q is $y^2 = 16a\ (x – 6x)$.

Solution

Let the coordinates of P and Q be $(at_1^2\ , 2at_1)$ and $(at_2^2\ , 2at_2)$ respectively.

As OP and OQ are perpendicular, we can have :

$$\left(\frac{2at_1 - 0}{at_1^2 - 0}\right)\left(\frac{2at_2 - 0}{at_2^2 - 0}\right) = -1$$

$\Rightarrow$ $t_1t_2 = -4$ ....(i)

Let the point of intersection of normals drawn at P and Q be $\equiv (x_1, y_1)$

Using the result given in section 1.4, we get :

$x_1 = 2a + a(t_1^2 + t_2^2 + t_1t_2\ )$ and ....(i)

$y_1 = –a\ t_1t_2\ (t_1 + t_2)$

Eliminating t_1 and t_2 from (i), (ii) and (iii), we get :

$y_1^2 = 16a\ (x_1 – 6a)$

The required locus is $y^2 = 16a\ (x – 6a)$

Example : 17

The normal at a point P to the parabola $y^2 = 4ax$ meets the X-axis in G. Show that P and G are equidistant from focus.

Solution

Let the coordinates of the point P be $(at^2, 2at)$

$\Rightarrow$ The equation of normal at P is : $tx + y = 2at + at^3$

The point of intersection of the normal with X-axis is $G \equiv (2a + at^2, 0)$.

$SP = a(1 + t^2)$ and $SG = \sqrt{(a + at^2)^2 + O^2} = a(1 + t^2)$.

$\Rightarrow$ SP = SG

Hence P and G are equidistant from focus.

Example : 18

Tangents to the parabola $y^2 = 4ax$ drawn at points whose abscise are in the ratio $\mu^2 : 1$. Prove that the locus of their point of intersection is $y^2 = [\mu 1/2 + \mu^{-1/2}]\ ax$.

Solution

Let the coordinates of the two points on which the tangents are drawn at $(at_1^2, 2at_1)$ and $(at_2^2, 2at_2)$.

As the abscissas are in the ratio $\mu^2 : 1$, we get :

$$\frac{at_1^2}{at_2^2} = \mu^2$$

$\Rightarrow$ $t_1 = \mu t_2$ ....(i)

Let the point of intersecting of two tangents be $M \equiv (x_1, y_1)$.

Using the result given in section 1.2, we get :

$M \equiv (x_1, y_1) \equiv [at_1\ t_2, a(t_1 + t_2)]$

$\Rightarrow$ $x_1 = at_1\ t_2$ ....(ii)

and $y_1 = a(t_1 + t_2)$ ....(iii)

Eliminate t_1 and t_2 from equations (i), (ii) and (iii) to get :

$y_1^2 = [\mu^{1/2} + \mu^{-1/2}]^2\ ax_1$

$\Rightarrow$ The required locus of M is : $y^2 = [\mu^{1/2} + \mu^{-1/2}]^2\ ax$.

Example : 19

Find the equation of common tangent to the circle $x^2 + y^2 = 8$ and parabola $y^2 = 16x$.

Solution

Let $ty = x + at^2$ (where a = 4) be a tangent to parabola which also touches circle.

$\Rightarrow$ $ty = x + 4t^2$ and $x^2 + y^2 = 8$ have only one common solution.

$\Rightarrow$ $(ty – 4t^2)^2 + y^2 = 8$ has equal roots as a quadratic in y.

⇒ $(1 + t^2) y^2 - 8t^3y + 16t^4 - 8 = 0$ has equal roots.

⇒ $64t^6 = 64t^6 + 64t^4 - 32 - 32t^2$

⇒ $t^2 + 1 - 2t^4 = 0$ ⇒ $t^2 = 1, -1/2$

⇒ $t = \pm 1$

⇒ the common tangents are $y = x + 4$ and $y = -x - 4$.

Example : 20

Through the vertex O of the parabola $y^2 = 4ax$, a perpendicular is drawn to any tangent meeting it at P and the parabola at Q. Show that OP. OQ = constant.

Solution

Let $ty = x + at^2$ be the equation of the tangent

OP = perpendicular distance of tangent from origin

⇒ $OP = -\dfrac{at^2}{\sqrt{1+r^2}}$

Equation of OP is $y - 0 = -t(x - 0)$ ⇒ $y = -tx$

Solving $y = -tx$ and $y^2 = 4ax$, we get

$$Q \equiv \left(\frac{4a}{t^2}, \frac{-4a}{t}\right)$$

⇒ $OQ^2 = \dfrac{16a^2}{t^4} + \dfrac{16a^2}{t^2}$

⇒ $OP \cdot OQ = 4a^2$

Example : 21

Prove that the circle drawn on any focal chord as diameter touches the directrix.

Solution

Let $P(t_1)$ and $Q(t_2)$ be the ends of a focal chord.

Using the result given in section 1.3, we get : $t_1t_2 = -1$

Equation of circle with PQ as diameter is :

$(x - at_1^2)(x - at_2^2) + (y - 2at_1)(y - 2at_2) = 0$ (using diametric form of equation of circle)

For the directrix to touch the above circle, equation of circle and directrix must have a unique solution i.e.

Solving $x = -a$ and circle simultaneously, we get

$a^2(1 + t^2)(1 + t_2^2) + y^2 - 2ay(t_1 + t_2) + 4a^2 t_1t_2 = 0$

This quadratic in y has discriminant = $D = B^2 - 4AC$

⇒ $D = 4a^2(t + t_2)^2 - 4a^2[(1 + t_1^2)(1 + t_1^2) + 4t_1t_2] = 0$ (using $t_1t_2 = -1$)

⇒ circle touches $x = -a$

⇒ circle touches the directrix.

Example : 22

Find the eccentricity, foci, latus rectum and directories of the ellipse $2x^2 + 3y^2 = 6$

Solution

The equation of the ellipse can be written as : $\dfrac{x^2}{3} + \dfrac{y^2}{2} = 1$

On comparing the above equation of ellipse with the standard equation of ellipse, we get

$a = \sqrt{3}$ and $b = \sqrt{2}$

We known that : $b^2 = a^2(1 - e^2)$

⇒ $2 = 3(1 - e^2)$ ⇒ $e = 1/\sqrt{3}$

Using the standard results, foci are (ae, 0) and (–ae, 0)

⇒ foci are (1, 0) and (–1, 0)

Latus rectum = $2b^2/a = 4\sqrt{3}$

Directrices are $x = \pm a/e$ ⇒ $x = \pm 3$

Example : 23

If the normal at a point P(θ) to the ellipse $\frac{x^2}{14} + \frac{y^2}{5} = 1$ intersect it again at Q(2θ), show that $\cos\theta = -2/3$.

Solution

The equation of normal at P(θ) : $\frac{ax}{\cos\theta} - \frac{by}{\sin\theta} = a^2 - b^2$

As Q ≡ (a cos 2θ, b sin 2θ) lies on it, we can have :

$$\frac{a}{\cos\theta}(a\cos 2\theta) - \frac{b}{\sin\theta}(b\sin 2\theta) = a^2 - b^2$$

$$\Rightarrow \quad a^2\,\frac{(2\cos^2\theta - 1)}{\cos\theta} - 2b^2\cos\theta = a^2 - b^2$$

Put $a^2 = 14$, $b^2 = 5$ in the above equation to get :

$14(2\cos^2\theta - 1) - 10\cos^2\theta = 9\cos\theta$

$\Rightarrow \quad 18\cos^2\theta - 9\cos\theta - 14 = 0$

$\Rightarrow \quad (6\cos\theta - 7)(3\cos\theta + 2) = 0$

$\Rightarrow \quad \cos\theta = 7/6$ (reject) or $\cos\theta = -2/3$

Hence $\cos\theta = -2/3$

Example : 24

If the normal at end of latus rectum passes through the opposite end of minor axis, find eccentricity.

Solution

The equation of the normal at L ≡ (ae, b^2/a) is given by :

$$\frac{a^2x}{ae} - \frac{b^2y}{b^2/a} = a^2 - b^2$$

$$\Rightarrow \quad \frac{x}{e} - y = \frac{a^2 - b^2}{a}$$

According to the question, B′ (0, –b) lies on the above normal.

$\Rightarrow \quad 0/e + b = (a^2 - b^2)/a$

$\Rightarrow \quad a^2 - b^2 - ab = 0$

Using $b^2 = a^2(1 - e^2)$, we get :

$a^2e^2 - ab = 0$

$\Rightarrow \quad b = ae^2$

$\Rightarrow \quad a^2e^4 = a^2(1 - e^2)$ [using : $b^2 = a^2(1 - e^2)$]

$\Rightarrow \quad e^4 = 1 - e^2$

$$\Rightarrow \quad e^2 - \frac{\sqrt{5} - 1}{2}$$

Example : 25

Show that the locus of the foot of the perpendicular drawn from the centre of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ on any tangent is $(x^2 + y^2) = a^2x^2 + b^2y^2$.

Solution

Let the tangent be $y = mx + \sqrt{a^2m^2 + b^2}$

Drawn CM is perpendicular to tangent and let M ≡ (x_1, y_1)

M lies on tangent, $\Rightarrow \quad y_1 = mx_1 + \sqrt{a^2m^2 + b^2}$ ....(i)

Slope (CM) = – 1/m

$\Rightarrow \quad \frac{y_1}{x_1} = -\frac{1}{m}$

$\Rightarrow \quad m = m = -\frac{x_1}{y_1}$ ....(ii)

Replace the value of m from (ii) into (i) to get :

$(x_1^2 + y_1^2)^2 = a^2 x_1^2 + b^2 y_1^2$

Hence the required locus is : $(x^2 + y^2)^2 = a^2 x^2 + b^2 y^2$

Example : 26

The tangent at a point P on ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ cuts the directrix in F. Show that PF subtends a right angle at the corresponding focus.

Solution

Let $P \equiv (x_1, y_1)$ and $S \equiv (ae, 0)$

The equation of tangent at P is : $\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1$

To find F, we put x = a/e in the equation of the tangent

$\Rightarrow \quad \frac{ax_1}{a^2 e} + \frac{yy_1}{b^2} = 1$

$\Rightarrow \quad y = \frac{(ae - x_1)b^2}{aey_1}$

$\Rightarrow \quad F \equiv \left[\frac{a}{e}, \frac{(ae - x_1)b^2}{aey_1}\right]$

$\Rightarrow \quad \text{slope (SF)} = \frac{(ae - x_1)b^2}{aey_1} \; \frac{1}{\frac{a}{e} - ae}$ ....(i)

$\text{slope (SP)} = \frac{y_1 - 0}{x_1 - ae}$ ....(ii)

From (i) and (ii),

slope of (SF) × slope (SP) = – 1

SF and SP are perpendicular

Hence PF subtends a right angle at the focus.

Example : 27

Show that the normal of ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ at any point P bisects the angle between focal radii SP and S′P.

Solution

Let PM be the normal and $P \equiv (x_1, y_1)$

$\Rightarrow \quad$ equation of normal PM is $\frac{xa^2}{x_1} - \frac{yb^2}{y_1} = a^2 - b^2$

We will try to show that : $\frac{S'P}{SP} = \frac{MS'}{MS}$

M is the point of intersection of normal PM with X-axis

$\Rightarrow$ Put y = 0 is normal PM to get M $\equiv \left[\frac{(a^2 - b^2)\, x_1}{a^2}, 0\right]$ = $[e^2x_1, 0]$

$\Rightarrow$ MS = ae – e^2x_1 and MS′ = ae = e^2x_1

$\Rightarrow$ $\frac{MS'}{MS} = \frac{e(a + ex_1)}{e(a - ex_1)} = \frac{a + ex_1}{a - ex_1} = \frac{SP'}{SP}$ (using result given in section 1.1)

Example : 28

A tangent to $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ touches at the point P on it in the first quadrant and meets the axes in A and B respectively. If P divides AB is 3 : 1, find the equation of tangent.

Solution

Let the coordinates of the point P ≡ (a cos θ , b sin θ)

$\Rightarrow$ the equation of the tangent at P is : $\frac{x\cos\theta}{a} + \frac{y\sin\theta}{b} = 1$ ....(i)

$\Rightarrow$ The coordinates of the points A and B are :

A $\equiv \left(\frac{a}{\cos\theta}, 0\right)$ and B $\equiv \left(0, \frac{b}{\sin\theta}\right)$

By section formula, the coordinates of P are $\left(\frac{a}{4\cos\theta}, \frac{3b}{3\sin\theta}\right)$ ≡ (a cos θ , b sin θ)

$\Rightarrow$ $\frac{a}{4\cos\theta}$ = a cos θ and $\frac{3b}{4\sin\theta}$ = b sin θ

$\Rightarrow$ cos θ = ± $\frac{1}{2}$ and sin θ = ± $\frac{\sqrt{3}}{2}$

$\Rightarrow$ θ = 60º

For equation of tangent, replace the value of θ in (i)

$\Rightarrow$ The equation of tangent is : $\frac{x}{a} + \frac{\sqrt{3}y}{b} = 2$

Example : 29

If the normal at point P of ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ with centre C meets major and minor axes at G and g respectively, and if CF be perpendicular to normal, prove that PF . PG = b^2 and PF . Pg = a^2.

Solution

If Pm is tangent to the ellipse at point P, then CMPF is a rectangle.

$\Rightarrow$ CM = PF ....(i)

Let the coordinates of point P be (a cos θ, b sin θ)

The equation of normal at P is : $\frac{ax}{\cos\theta} - \frac{by}{\sin\theta} = a^2 - b^2$

The point of intersection of the normal at P with X-axis is G $\equiv \left[\frac{(a^2 - b^2)\cos\theta}{a}, 0\right]$

The point of intersection of the normal at P with Y-axis is g $\equiv \left[0, \frac{(b^2 - a^2)\sin\theta}{b}\right]$

$\Rightarrow$ $PG^2 = \frac{b^2}{a^2}\,[b^2\cos^2\theta + a^2\sin^2\theta]$ ....(ii)

and $Pg^2 = \frac{a^2}{b^2}\,[b^2\cos^2\theta + a^2\sin^2\theta]$ ....(iii)

From (i),

$\Rightarrow$ PF = MC = distance of centre of the ellipse from the tangent at P

$$= \frac{1}{\sqrt{\frac{\cos^2\theta}{a^2} + \frac{\sin^2\theta}{b^2}}} = \frac{ab}{\sqrt{b^2\cos^2 + a^2\sin^2\theta}} \quad \text{...........(iv)}$$

Multiplying (iii) and (iv), we get :

$PF^2 \, . \, PG^2 = b^4$

Multiplying (iii) and (iv), we get :

$PF^2 \, . \, Pg^2 = a^4$

Hence proved

Example : 30

Any tangent to an ellipse is cut by the tangents at the ends of the major axis in T and T′. Prove that circle on TT′ as diameter passes through foci.

Solution

Consider a point P on the ellipse whose coordinates are (a cos θ , b sin θ)

The equation of tangent drawn at P is : $\frac{x\cos\theta}{a} + \frac{y\sin\theta}{b} = 1$ ....(i)

The two tangents drawn at the ends of the major axis are x = a and x = – a.

Solving tangent (i) and x = a we get T = $\left[a, \frac{b(1-\cos\theta)}{\sin\theta}\right] \equiv \left[a, b\tan\frac{\theta}{2}\right]$

Solving tangent (i) and x = – a, we get T′ ≡ $\left[-a, \frac{b(1+\cos\theta)}{\sin\theta}\right] = \left[-a, \cot\frac{\theta}{2}\right]$

Circle on TT′ as diameter is $x^2 - a^2 + (y - b\tan\theta/2)(y - b\cot\theta/2) = 0$

(using diametric form of equation of circle)

Put x = ± ae, y = 0 in LHS to get :

$a^2e^2 - a^2 + b^2 = 0$ = RHS

Hence foci lie on this circle.

Example : 31

A normal inclined at 45º to the X-axis is drawn to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. It cuts major and minor axes in P and Q. If C is centre of ellipse, show that are (ΔCPQ) = $\frac{(a^2-b^2)^2}{a(a^2+b^2)}$.

Solution

Consider a point M on the ellipse whose coordinates are (a cos θ, b sin θ)

The equation of normal drawn at M is : $\frac{ax}{\cos\theta} - \frac{by}{\sin\theta} = a^2 - b^2$

As the normal makes an angle 45º with X-axis, slope of normal = tan 45º

$\Rightarrow$ $\tan 45^\circ = \frac{a\sin\theta}{b\cos\theta}$ $\Rightarrow$ $\tan\theta = \frac{b}{a}$

$\Rightarrow \quad \sin\theta = \dfrac{b}{\sqrt{a^2+b^2}}$ and $\cos\theta = \dfrac{a}{\sqrt{a^2+b^2}}$ ....(i)

The point of intersecting of the normal with X-axis is $P \equiv \left[\dfrac{a^2-b^2}{a}\cos\theta, 0\right]$

$\Rightarrow \quad CP = \left|\dfrac{a^2-b^2}{a}\cos\theta\right|$ ....(ii)

The point of intersection of the normal with Y-axis is $Q \equiv \left[0, \dfrac{b^2-a^2}{b}\sin\theta\right]$

$\Rightarrow \quad CQ = \left|\dfrac{b^2-a^2}{b}\right|$ ....(iii)

Ar (ΔCPQ) = $\dfrac{1}{2}$ PC × CQ

Using (ii) and (iii), Ar (ΔCPQ) = $\dfrac{1}{2}\left|\dfrac{(a^2-b^2)^2}{ab}\sin\theta\cos\theta\right|$

Using (i), Ar (ΔCPQ) = $\dfrac{1}{2}\dfrac{(a^2-b^2)^2}{a^2+b^2}$

Example : 32

If P, Q are points on $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$, whose centre is C such that CP is perpendicular to CQ, show that

$\dfrac{1}{CP^2} + \dfrac{1}{CQ^2} = \dfrac{1}{a^2} - \dfrac{1}{b^2}$ given that (a < b)

Solution

Let y = mx be the equation of CP. Solving y = mx and $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$, we get coordinates of P.

$\Rightarrow \quad \dfrac{x^2}{a^2} - \dfrac{m^2x^2}{b^2} = 1 \quad \Rightarrow \quad x^2 = \dfrac{a^2b^2}{b^2-a^2m^2}, \; y^2 = \dfrac{a^2b^2m^2}{b^2-a^2m^2}$

$\Rightarrow \quad CP^2 = x^2 + y^2 = \dfrac{a^2b^2(1+m^2)}{b^2-a^2m^2}$

Similarly, be replacing m by – 1/m, we get coordinates of Q because equation of CQ is $y = \dfrac{-1}{m}x$.

$\Rightarrow \quad CQ^2 = \dfrac{a^2b^2\left(1+\dfrac{1}{m^2}\right)}{b^2-\dfrac{a^2}{m^2}} = \dfrac{a^2b^2(m^2+1)}{b^2m^2-a^2}$

$\Rightarrow \quad \dfrac{1}{CP^2} + \dfrac{1}{CQ^2} = \dfrac{b^2-a^2m^2+b^2m^2-a^2}{a^2b^2(1+m^2)} = \dfrac{b^2-a^2}{a^2b^2} = \dfrac{1}{a^2} - \dfrac{1}{b^2}$

Example : 33

Find the locus of the foot of the perpendicular drawn from focus S of hyperbola $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ to any tangent.

Solution

Let the tangent be $y = mx + \sqrt{a^2m^2 - b^2}$

Let m (x_1, y_1) be the foot of perpendicular SM drawn to the tangent from focus S (ae, 0).

Slope (SM) × slope (PM) = – 1

$\Rightarrow \quad \left(\frac{y_1 - 0}{x_1 - ae}\right) m = -1$

$x_1 + my_1 = ae$ ....(i)

As M lies on tangent, we also have $y_1 = m_1x + \sqrt{a^2m^2 - b^2}$

$\Rightarrow \quad -mx_1 + y_1 = \sqrt{a^2m^2 - b^2}$ ....(ii)

We can now eliminate m from (i) and (ii).

Substituting value of m from (i) in (ii) leads to a lot of simplification and hence we avoid this step.

By squaring and adding (i) and (ii), we get

$x_1^2 (1 + m^2) + y_1^2 (1 + m^2) = a^2e^2 + a^2m^2 - b^2$

$\Rightarrow \quad (x_1^2 + y_1^2) (1 + m^2) = a^2 (1 + m^2)$

$\Rightarrow \quad x_1^2 + y_1^2 = a^2$

$\Rightarrow$ Required locus is : $x^2 + y^2 = a^2$ (Note that M lies on the auxiliary circle)

Example : 34

Prove that the portion of the tangent to the hyperbola intercepted between the asymptotes is bisected at the point of contact and the area of the triangle formed by the tangent and asymptotes is constant.

Solution

Let $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ be the hyperbola and let the point of contact be P $(a \sec\theta, b \tan\theta)$

Let the tangent meets the asymptotes $y = \frac{bx}{a}$ and $y = -\frac{bx}{b}$ in points M, N respectively.

Solving the equation of tangent and asymptotes, we can find M and N

Solve : $\frac{x\cos\theta}{a} - \frac{y\tan\theta}{b} = 1$ and $y = \frac{bx}{a}$ to get :

$x = \frac{a}{\sec\theta - \tan\theta}, \quad y = \frac{b}{\sec\theta - \tan\theta}$

$\Rightarrow \quad M \equiv \left[\frac{a}{\sec\theta - \tan\theta}, \frac{b}{\sec\theta - \tan\theta}\right],$

Similarly solving $y = -\frac{bx}{a}$ and $\frac{x}{a} \sec\theta - \frac{y}{a} \tan\theta = 1$, we get :

$N \equiv \left[\frac{a}{\sec\theta + \tan\theta}, \frac{-b}{\sec\theta + \tan\theta}\right]$

Mid point of MN $\equiv \left[\frac{a\sec\theta}{\sec^2\theta - \tan^2\theta}, \frac{b\tan\theta}{\sec^2\theta - \tan^2\theta}\right] \equiv (a \sec\theta, b \tan\theta)$

Hence P bisects MN.

Area of $\Delta CNM = \frac{1}{2}\begin{vmatrix} 0 & 0 & 1 \\ x_N & y_N & 1 \\ x_M & y_M & 1 \end{vmatrix} = \frac{1}{2}\{x_N y_M - x_M y_N\} = \frac{1}{2}(ab + ab) = ab$

hence area does not depend on 'θ' or we can say that area is constant.

Example : 35

Show that the locus of the mid-point of normal chords of the rectangular hyperbola $x^2 - y^2 = a^2$ is $(y^2 - x^2)^3 = 4a^2x^2y^2$.

Solution

Let the mid point of a chord be $P(x_1, y_1)$

$\Rightarrow$ Equation of chord of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ whose mid-point is (x_1, y_1) is :

$$\frac{xx_1}{a^2} - \frac{yy_1}{b^2} = \frac{x_1^2 - y_1^2}{a^2}$$

As the hyperbola is rectangular hyperbola, a = b

$\Rightarrow$ Equation of the chord is : $xx_1 - yy_1 = x_1^2 - y_1^2$ ....(i)

Normal chord is a chord which is normal to hyperbola at one of its ends.

$\Rightarrow$ Equation of normal chord at $(a \sec\theta, b \tan\theta)$ is : $\frac{ax}{\sec\theta} - \frac{by_1}{\tan\theta} = a^2 + b^2$

but here $a^2 = b^2$,

$\Rightarrow$ normal chord is : $x \cos\theta - y \cot\theta = 2a$ ....(ii)

We now compare the two equations of same chord i.e. compare (i) and (ii) to get :

$$\Rightarrow \quad \frac{x_1}{\cos\theta} = \frac{y_1}{\cot\theta} = \frac{x_1^2 - y_1^2}{2a}$$

$$\Rightarrow \quad \sec\theta = \frac{x_1^2 - y_1^2}{2ax_1} \quad \text{and} \quad \cot\theta = \frac{2ay_1}{x_1^2 - y_1^2}$$

Eliminating θ using $\sec^2\theta - \tan^2\theta = 1$, we get :

$$\left(\frac{x_1^2 - y_1^2}{2ax_1}\right)^2 - \left(\frac{x_1^2 - y_1^2}{2ay_1}\right)^2 = 1$$

$\Rightarrow$ $(y_1^2 - x_1^2)^3 = 4a^2 x_1^2 y_1^2$

$\Rightarrow$ $(y^2 - x^2)^3 = 4a^2x^2y^2$ is the locus.