Example : 1

Evaluate (i) $\int_1^3 x^2\,dx$ (ii) $\int_0^{\pi/2} \sin x\,dx$

Solution

(i) $\int_1^3 x^2\,dx = \left|\frac{x^3}{3}\right|_1^3 = \frac{1}{3}\,(3^3 - 1^3) = \frac{26}{3}$

(ii) $\int_0^{\pi/2} \sin x\,dx = \left|-\cos x\right|_0^{\pi/2} = (\cos \pi/2 - \cos 0) = 1$

Example : 2

$\int_0^{\pi/2} \sin^3 x \cos x\,dx$

Solution

Let $I = \int_0^{\pi/2} \sin^3 x \cos x\,dx$

Let $\sin x = t \Rightarrow \cos x\,dx = dt$

For $x = \frac{\pi}{2}$, $t = 1$ and for $x = 0$, $t = 0$

$\Rightarrow I = \int_0^1 t^3\,dt = \left|\frac{t^4}{4}\right|_0^1 = \frac{1}{4}$

Note : Whenever we use substitution in a definite integral, we have to change the limits corresponding to the change in the variable of the integration

In the example we have applied New-ton-Leibnitz formula to calculate the definite integral. New-Leibnitz formula is applicable here since $\sin^3 x \cos x$ (integrate) is a continuous function in the interval $[0, \pi/2]$

Example : 3

Evaluate : $\int_{-1}^{2} |x|\,dx$

Solution

$\int_{-1}^{2} |x|\,dx = \int_{-1}^{0} |x|\,dx + \int_{0}^{2} |x|\,dx$ (using property – 1)

$= \int_{-1}^{0} -x\,dx + \int_{0}^{2} x\,dx$ ($\because |x| = -x$ for $x < 0$ and $|x| = x$ for $x \ge 0$)

$= -\left|\frac{x^2}{2}\right|_{-1}^{0} + \left|\frac{x^2}{2}\right|_{0}^{2} = -\left(0 - \frac{1}{2}\right) + \left(\frac{4}{2} - 0\right) = \frac{5}{2}$

Example : 4

Evaluate : $\int_{-4}^{3} |x^2 - 4|\, dx$

Solution

$$\int_{-4}^{3} |x^2 - 4|\, dx = \int_{-4}^{-2} |x^2 - 4|\, dx + \int_{-2}^{+2} |x^2 - 4|\, dx + \int_{-2}^{3} |x^2 - 4|\, dx$$

$$= \int_{-4}^{-2} (x^2 - 4)\, dx + \int_{-2}^{2} (4 - x^2)\, dx + \int_{2}^{3} (x^2 - 4)\, dx$$

($\because$ $|x^2 - 4| = 4 - x^2$ in [–2, 2] and $|x^2 - 4| = x^2 - 4$ in other intervals]

$$= \left|\frac{x^3}{3} - 4x\right|_{-4}^{-2} + \left|4x - \frac{x^3}{3}\right|_{-2}^{2} + \left|\frac{x^3}{3} - 4x\right|_{2}^{3}$$

$$= \left(-\frac{8}{3} + 8\right) - \left(\frac{64}{3} + 16\right) + \left(8 - \frac{8}{3}\right) + \left(\frac{27}{3} - 12\right) - \left(\frac{8}{3} - 8\right) = \frac{71}{3}$$

Example : 5

Evaluate : $\int_{0}^{\pi/2} \frac{\sqrt{\sin x}}{\sqrt{\sin x} + \sqrt{\cos x}}\, dx$

Solution

Let $I = \int_{0}^{\pi/2} \frac{\sqrt{\sin x}}{\sqrt{\sin x} + \sqrt{\cos x}}\, dx$ ....(i)

Using property – 4, we have :

$$I = \int_{0}^{\pi/2} \frac{\sqrt{\sin(\pi/2 - x)}}{\sqrt{\sin(\pi/2 - x)} + \sqrt{\cos(\pi/2 - x)}}$$

$$I = \int_{0}^{\pi/2} \frac{\sqrt{\cos x}}{\sqrt{\cos x} + \sqrt{\sin x}}\, dx \quad \text{...........(ii)}$$

Adding (i) and (ii), we get

$$2I = \int_{0}^{\pi/2} \frac{\sqrt{\sin x}}{\sqrt{\sin x} + \sqrt{\cos x}}\, dx + \int_{0}^{\pi/2} \frac{\sqrt{\cos x}}{\sqrt{\sin x} + \sqrt{\cos x}}\, dx$$

$$\Rightarrow \quad 2I = \int_{0}^{\pi/2} \frac{\sqrt{\sin x} + \sqrt{\cos x}}{\sqrt{\sin x} + \sqrt{\cos x}}\, dx$$

$$\Rightarrow \quad 2I = \int_{0}^{\pi/2} dx = \frac{\pi}{2}$$

$$\Rightarrow \quad I = \frac{\pi}{4}$$

Example : 6

If f(a – x) = f(x), then show that $\int_0^a x\, f(x)\, dx = \frac{a}{2}\int_0^a f(x)\, dx$

Solution

Let $I = \int_0^a x\, f(x)\, dx$

$\Rightarrow \quad I = \int_0^a (a - x)\, f\,(a - x)\, dx$ (using property – 4)

$\Rightarrow \quad I = \int_0^a (a - x)\, f\,(x)\, dx$ [using f(x) = f(a – x)]

$\Rightarrow \quad I = \int_0^a a\, f(x)\, dx - \int_0^a x\, f(x)\, dx$

$\Rightarrow \quad I = a\int_0^a f(x)\, dx - I$

$\Rightarrow \quad 2I = a\int_0^a f(x)\, dx$

$\Rightarrow \quad I = \frac{a}{2}\int_0^a f(x)\, dx = \text{RHS}$

Example : 7

Evaluate : $\int_0^{\pi} \frac{x}{1+\cos^2 x}\, dx$

Solution

Let $I = \int_0^{\pi} \frac{x}{1+\cos^2 x}\, dx$ ....(i)

$\Rightarrow \quad I = \int_0^{\pi} \frac{(\pi - x)}{1+\cos^2(\pi - x)}\, dx$ (using property – 4) ....(ii)

Adding (i) and (ii), we get :

$\Rightarrow \quad 2I = \int_0^{\pi} \frac{\pi}{1+\cos^2 x}\, dx$

$\Rightarrow \quad I = \frac{\pi}{2}\int_0^{\pi} \frac{dx}{1+\cos^2 x} = \frac{2\pi}{2}\int_0^{\pi/2} \frac{dx}{1+\cos^2 x}$ (using property – 6)

Divide N and D by $\cos^2 x$ to get :

$$I = \int_0^{\pi/2} \frac{\sec^2 x}{\sec^2 x + 1}\, dx$$

Put tan x = t $\Rightarrow$ $\sec^2 x\, dx = dt$

For x = π/2, $t \to \infty$ and for x = 0, t = 0

$$\Rightarrow \quad I = \pi \int_0^{\infty} \frac{dt}{2 + t^2}$$

$$\Rightarrow \quad I = \left| \frac{1}{\sqrt{2}} \tan^{-1} \frac{t}{\sqrt{2}} \right|_0^{\infty} = \frac{\pi}{\sqrt{2}} \times \frac{\pi}{2} = \frac{\pi^2}{2\sqrt{2}}$$

Example : 8

Evaluate : $\int_0^{\pi/2} \ell og \sin x\, dx$

Solution

Let $I = \int_0^{\pi/2} \ell og \sin x\, dx$ ....(i)

$\Rightarrow \quad I = \int_0^{\pi/2} \ell og \sin\left(\frac{\pi}{2} - x\right) dx$ (using property – 4)

$\Rightarrow \quad I = \int_0^{\pi/2} \ell og \cos x\, dx$ ....(ii)

Adding (i) and (ii) we get :

$$2I = \int_0^{\pi/2} \ell og (\sin x \cos x)\, dx = \int_0^{\pi/2} \ell og \left(\frac{\sin 2x}{2}\right) dx$$

$$\Rightarrow \quad 2I = \int_0^{\pi/2} \ell og \sin 2x\, dx - \int_0^{\pi/2} \log 2\, dx$$

$\Rightarrow \quad 2I = \int_0^{\pi/2} \ell og \sin 2x\, dx - \frac{\pi}{2} \ell og 2$ ....(iii)

Let $I_1 = \int_0^{\pi/2} \ell og \sin 2x\, dx$

Put t = 2x $\Rightarrow$ dt = 2dx

For $x = \frac{\pi}{2}$, t = π and for x = 0, t = 0

$\Rightarrow \quad I_1 = \frac{1}{2} \int_0^{\pi} \log \sin t\, dt = \frac{2}{2} \int_0^{\pi/2} \log \sin t\, dt$ (using property – 6)

$\Rightarrow \quad I_1 = \int_0^{\pi/2} \log \sin x\, dx$ (using property – 3)

$\Rightarrow \quad I_1 = I$

Substituting in (iii), we get : $2I = I - \pi/2 \log 2$

$\Rightarrow \quad I = -\pi/2 \log 2$ (learn this result so that you can directly apply it in other difficult problems)

Example : 9

Show that : $\int_0^{\pi/2} f(\sin 2x) \sin x \, dx = \int_0^{\pi/2} f(\cos x \, dx = \sqrt{2} \int_0^{\pi/4} f(\cos 2x) \cos x \, dx$

Solution

Let $I = \int_0^{\pi/2} f(\sin 2x) \sin x \, dx$ ....(i)

$\Rightarrow \quad I = \int_0^{\pi/2} f[\sin 2(\pi/2 - x)] \sin(\pi/2 - x) \, dx$ (using property – 4)

$\Rightarrow \quad I = \int_0^{\pi/2} f[\sin(\pi - 2x)] \cos x \, dx$

$\Rightarrow \quad I = \int_0^{\pi/2} f(\sin 2x) \cos x \, dx$ ....(ii)

Hence the first part is proved

$I = \int_0^{\pi/2} f(\sin 2x) \sin x \, dx$

$= \int_0^{\pi/4} f(\sin 2x) \sin x \, dx + \int_0^{\pi/4} f[\sin 2(\pi/2 - x)] \sin(\pi/2 - x) \, dx$ (using property – 5)

$= \int_0^{\pi/4} f(\sin 2x) \sin x \, dx + \int_0^{\pi/4} f(\sin 2x) \cos x \, dx$

$= \int_0^{\pi/4} f(\sin 2x) (\sin x + \cos x) \, dx$

$= \int_0^{\pi/4} f(\sin 2x) (\sin x + \cos x) \, dx$

$= \int_0^{\pi/4} f[\sin 2(\pi/4 - x)] [\sin(\pi/4 - x) + \cos(\pi/4 - x) x] \, dx$ (using property – 4)

$= \int_0^{\pi/4} f(\cos 2x) \left[\frac{1}{\sqrt{2}} \cos x - \frac{1}{\sqrt{2}} \sin x + \frac{1}{\sqrt{2}} \cos x + \frac{1}{\sqrt{2}} \sin x\right] dx$

$= \sqrt{2} \int_0^{\pi/4} f(\cos 2x) \cos x \, dx$

Hence the second part is also proved

Example : 10

Evaluate : $\int_2^3 x\sqrt{5-x}\, dx$

Solution

Let $I = \int_2^3 x\sqrt{5-x}\, dx$

$\Rightarrow \quad I = \int_2^3 (2+3-x)\sqrt{5-(2+3-x)}\, dx$ (using property – 7)

$\Rightarrow \quad I = \int_2^3 (5-x)\sqrt{x}\, dx$

$\Rightarrow \quad I = \int_2^3 5\sqrt{x}\, dx - \int_2^3 x\sqrt{x}\, dx$

$\Rightarrow \quad I = 5\left|\frac{2}{3}x\sqrt{x}\right|_2^3 - \frac{2}{5}\left|x^2\sqrt{x}\right|_2^3$

$\Rightarrow \quad I = \frac{10}{3}\left(3\sqrt{3}-2\sqrt{2}\right) - \frac{2}{5}\left(9\sqrt{3}-4\sqrt{2}\right)$

Example : 11

Evaluate : $\int_a^b \frac{f(x)}{f(x)+f(a+b+x)}\, dx$

Solution

Let $I = \int_a^b \frac{f(x)}{f(x)+f(a+b-x)}\, dx$ ....(i)

$\Rightarrow \quad I = \int_a^b \frac{f(a+b-x)}{f(a+b-x)+f[a+b-(a+b-x)]\, dx}$

$\Rightarrow \quad I = \int_a^b \frac{f(a+b-x)}{f(a+b-x)+f(x)}\, dx$ ....(ii)

Adding (i) and (ii), we get

$\Rightarrow \quad 2I = \int_a^b \frac{f(x)+f(a+b-x)}{f(x)+f(a+b-x)}\, dx$

$\Rightarrow \quad 2I = \int_a^b dx = b-a$

$\Rightarrow \quad I = \frac{b-a}{2}$

Example : 12

Evaluate : $\int_{-1}^{+1} \log\left(\frac{2-x}{2+x}\right) \sin^2 x \, dx$

Solution

Let $f(x) = \log\left(\frac{2-x}{2+x}\right) \sin^2 x \, dx$

$\Rightarrow \quad f(-x) = \log\left(\frac{2+x}{2-x}\right) \sin^2 (-x)$

$\Rightarrow \quad f(-x) = \log\left(\frac{2-x}{2+x}\right)^{-1} \sin^2 x = -\log\left(\frac{2-x}{2+x}\right) \sin^2 x = -f(x)$

$\Rightarrow \quad$ f(x) is an odd function

Hence $\int_{-1}^{+1} f(x)\, dx = 0$ (using property – 8)

Example : 13

Evaluate : $\int_0^{\pi/2} \sqrt{1-\sin 2x} \; dx$

Solution

Let $\quad I = \int_0^{\pi/2} \sqrt{1-\sin 2x} \; dx$

$\Rightarrow \quad I = \int_0^{\pi/2} \sqrt{(\sin x - \cos x)^2} \; dx$

$\Rightarrow \quad I = \int_0^{\pi/2} |\sin x - \cos x| \; dx$

$\Rightarrow \quad I = \int_0^{\pi/4} |\sin x - \cos x| \, dx + \int_{\pi/4}^{\pi/2} |\sin x - \cos x| \, dx$

$\Rightarrow \quad I = \int_0^{\pi/4} (\cos x - \sin x) \, dx + \int_{\pi/4}^{\pi/2} (\sin x - \cos x) \, dx$

$\Rightarrow \quad I = |\sin x + \cos|_0^{\pi/4} + | -\cos x - \sin x |_{\pi/4}^{\pi/2}$

$\Rightarrow \quad I = \left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} - 1\right) + (-1) - \left(-\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}\right)$

$\Rightarrow \quad I = 2\sqrt{2} - 2$

Example : 14

Given a function such that :

(i) it is integrable over every interval on the real line.

(ii) f(t + x) = f(x) for every x and a real t, then show that the integral $\int_a^{a+t} f(x)\,dx$ is independent of a.

Solution

Let $I = \int_a^{a+t} f(x)\,dx$

$\Rightarrow \int_a^{t} f(x)\,dx + \int_t^{a+t} f(x)\,dx$ ....(i)

Consider $I_1 = \int_t^{a+t} f(x)\,dx$

Put x = y + t $\Rightarrow$ dx = dy

For x = a + t, y = a and For x = t, y = 0

$\Rightarrow \quad I_1 = \int_0^{a} f(y+t)\,dy$

$\Rightarrow \quad I_1 = \int_0^{a} f(y)\,dy$ (using property 3)

$\Rightarrow \quad I_1 = \int_0^{a} f(x)\,dx$ [using f(x + T) = f(x)]

On substituting the value of I_1 in (i), we get :

$\Rightarrow \quad I = \int_a^{t} f(x)\,dx + I_1$

$\Rightarrow \quad I = \int_a^{t} f(x)\,dx + \int_0^{a} f(x)\,dx$

$\Rightarrow \quad I = \int_0^{a} f(x)\,dx + \int_a^{t} f(x)\,dx$

$\Rightarrow \quad I = \int_0^{t} f(x)\,dx$ (using property – 1)

$\Rightarrow$ I is independent of a.

Example : 18

Determine a positive integer $n \le 5$ such that : $\int_0^1 e^x (x-1)^n dx = 16 - 6e$

Solution

Let $I_n = \int_0^1 e^x (x-1)^n dx$

using integration by parts

$I_n = \left[(x-1)^n \int e^x\, dx\right]_0^1 - \int_0^1 e^x n(x-1)^{n-1}\, dx$

$I_n = 0 - (-1)^n - n \int_0^1 e^x (x-1)^{n-1}\, dx$

$I_n = -(-1)^n - nI_{n-1}$ ....(i)

Also $I_0 = \int_0^t e^x (x-1)^0\, dx = e - 1$

$\Rightarrow$ $I_1 = 1 - I_0 = 1 - (e - 1) = 2 - e$
$\Rightarrow$ $I_2 = -1 - 2I_1 = -1 - 2(2 - e) = -5 + 2e$
$\Rightarrow$ $I_3 = 1 - 3I_2 = 1 - 3(-5 + 2e) = 16 - 6e$

$\Rightarrow$ Hence for n = 3, $\int_0^1 e^x (x-1)^n\, dx = 16 - 6e$

Example : 16

If $f(x) = \int_{x^2}^{x^3} \frac{1}{\log t} dt$ $t > 0$, then find $f'(x)$

Solution

Using the property – 12,

$$f'(x) = \frac{1}{\log(x^3)} \frac{d}{dx}(x^3) + \frac{1}{\log x^2} \frac{d}{dx}(x^2)$$

$$\Rightarrow \quad f'(x) = \frac{3x^2}{3\log x} - \frac{2x}{2\log x} = \frac{x^2 - x}{\log x}$$

Example : 17

Find the points of local minimum and local minimum of the function $\int_0^{x^2} \frac{t^2 - 5t + 4}{2 + e^t} dt$

Solution

Let $y = \int_0^{x^2} \frac{t^2 - 5t + 4}{2 + e^t} dt$

For the points of Extremes, $\frac{dy}{dx} = 0$

Using property – 12

$$\frac{x^4 - 5x^2 + 5}{2 + e^{x^2}}\ 2x = 0$$

⇒ $x = 0$ or $x^4 - 5x^2 + 4 = 0$

⇒ $x = 0$ or $(x - 1)(x + 1)(x - 2)(x + 2) = 0$

⇒ $x = 0$, $x = \pm 1$ and $x = \pm 2$

With the help of first derivative test, check yourself x = – 2, 0, 2 are points of local minimum and x = – 1, 1 are points of local maximum.

Example : 18

Evaluate : $\int_a^b x^2 dx$ using limit of a sum formula

Solution

$$\text{Let } I = \int_a^b x^2 dx = \lim_{\substack{n\to\infty \\ h\to 0}} h\,[1 + h)^2 + (1 + 2h)^2 + \ldots\ldots.. + (a + nh)^2\,]$$

$$\Rightarrow \quad I = \lim_{\substack{n\to\infty \\ h\to 0}} h[na^2 + 2ah\,(1 + 2 + 3 + \ldots\ldots. + n) + h^2\,(1^2 + 2^2 + 3^2 + \ldots\ldots.+n^2)]$$

$$\Rightarrow \quad I = \lim_{\substack{n\to\infty \\ h\to 0}} \left[nha^2 + \frac{2ah^2 n(n+1)}{2} + \frac{h^2 n(n+1)(2n+1)}{6}\right]$$

Using nh = b – a, we get

$$\Rightarrow \quad I = \lim_{n\to\infty} \left[a^2(b-a) + a(b-a)^2\left(1+\frac{1}{n}\right) + (b-a)^3\,\frac{1}{6}\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right)\right]$$

$$\Rightarrow \quad I = a^2\,(b - a) + a\,(b - a)^2 + \frac{(b-a)^2}{6}\ 2$$

$$\Rightarrow \quad I = (b - a)\left[a^2 + ab - a^2 + \frac{b^2 + a^2 - 2ab}{3}\right]$$

$$\Rightarrow \quad I = \frac{(b-a)}{3}\,[a^2 + b^2 + ab] = \frac{b^3 - a^3}{3}$$

Example : 19

Evaluate the following sum. $S = \lim_{n\to\infty} \left[\frac{1}{n+1} + \frac{1}{n+2} + \frac{1}{n+3} + \ldots.. + \frac{1}{2n}\right]$

Solution

$$S = \lim_{n\to\infty} \left[\frac{1}{n+1} + \frac{1}{n+2} + \frac{1}{n+3} + \ldots.. + \frac{1}{2n}\right]$$

$$\Rightarrow \quad S = \lim_{n\to\infty} \frac{1}{n}\left[\frac{n}{n+1} + \frac{n}{n+2} + \ldots. + \frac{n}{2n}\right]$$

$$\Rightarrow \quad S = \lim_{n\to\infty} \frac{1}{n}\left[\frac{1}{1+1/n} + \frac{1}{1+2/n} + \ldots\ldots. + \frac{1}{1+n/n}\right]$$

$$\Rightarrow \quad S = \lim_{n\to\infty} \frac{1}{n}\left[\sum_{r=1}^{n} \frac{1}{1+r/n}\right]$$

$$\Rightarrow \quad \int_0^1 \frac{1}{1+x}\,dx$$

$$\Rightarrow \quad S = \left|\log(1+x)\right|_0^1 = \log 2$$

Example : 20

Find the sum of the series : $\lim_{n\to\infty} \frac{1}{n} + \frac{1}{n+1} + \frac{1}{n+2} + \ldots\ldots\ldots + \frac{1}{6n}$

Solution

Let $S = \lim_{n\to\infty} \frac{1}{n} + \frac{1}{n+1} + \frac{1}{n+2} + \ldots\ldots\ldots + \frac{1}{6n}$

Take 1/n common from the series i.e.

$$S = \lim_{n\to\infty} \frac{1}{n}\left[\frac{1}{1+1/n} + \frac{1}{1+2/n} + \ldots\ldots\ldots + \frac{1}{1+5n/n}\right] = \lim_{n\to\infty} \frac{1}{n} \sum_{r=0}^{2n} \frac{1}{1+r/n}$$

For the definite integral,

$$\text{Lower limit} = a = \lim_{n\to\infty}\left(\frac{r}{n}\right) = \lim_{n\to\infty} \frac{1}{n} = 0$$

$$\text{Upper limit} = b = \lim_{n\to\infty}\left(\frac{r}{n}\right) = \lim_{n\to\infty} \frac{5n}{n} = 5$$

Therefore, $$S = \lim_{n\to\infty} \sum_{r=0}^{5n} \frac{1}{1+(r/n)} = \int_0^5 \frac{dx}{1+x} = \ell n \left|1+x\right|_0^5 = \ell n = 6 - \ell n\ 1 = \ell n\ 6$$

Example : 21

Show that : $$1 \le \int_0^1 e^{x^2} dx \le e$$

Solution

Using the result given in section 3.3,

$$m\,(1-0) \le \int_0^1 e^{x^2} dx \le M(1-0) \qquad \ldots\ldots\ldots.(i)$$

let $\quad f(x) = e^{x^2}$

$\Rightarrow \quad f'(x) = 2x\,e^{x^2} = 0 \qquad \Rightarrow \qquad x = 0$

Apply first derivative test to check that there exists a local minimum at x = 0

$\Rightarrow \quad$ f(x) is an increasing function in the interval [0, 1]

$\Rightarrow \quad m = f(0) = 1 \quad$ and $\quad M = f(1) = e^1 = e$

Substituting the value of m and M in (i), we get

$$(1-0) \le \int_0^1 e^{x^2} dx \le e(1-0)$$

$$\Rightarrow \quad 1 \le \int_0^1 e^{x^2} dx \le e$$ Hence proved.

Example : 22

Consider the integral : $I = \int_0^{2\pi} \frac{dx}{5 - 2\cos x}$

Making the substitution tan x/2 = t, we have :

$$\int_0^{2\pi} \frac{dx}{5 - 2\cos x} = \int_0^0 \frac{2dt}{(1+t^2)\left[5 - 2\frac{1-t^2}{1+t^2}\right]} = 0$$

This result is obviously wrong since the integrand is positive and consequently the integral of this function can not be equal to zero. Find the mistake in this evaluation.

Solution

The mistake lies in the substitution $\tan\frac{x}{2} = t$. Since the function $\tan\frac{x}{2}$ is discontinuous at $x = \pi$, a point in the interval $(0, 2\pi)$, we can not use this substitution for the changing the variable of integration.

Example : 23

Find the mistake in the following evaluation of the integral

$$\int_0^{\pi} \frac{dx}{1+2\sin^2 x} = \int_0^{\pi} \frac{dx}{\cos^2 x + 3\sin^2 x} = \int_0^{\pi} \frac{\sec^2 x\, dx}{1+3\sin^2 x} = \frac{1}{\sqrt{3}}\left[\tan^{-1}\left(\sqrt{3}\tan x\right)\right]_0^{\pi} = 0$$

Solution

The Newton-Leibnitz formula for evaluating the definite integrals is not applicable here since the anti-derivative.

$F(x) = \frac{1}{\sqrt{3}}\tan^{-1}\left(\sqrt{3}\tan x\right)$ has a discontinuity at the point $x = \pi/2$ which lies in the interval $[0, \pi]$.

$$\underset{\text{at } x=\pi/2}{\text{LHL}} = \lim_{h\to 0} \frac{1}{\sqrt{3}}\tan^{-1}\left[\tan\left(\frac{\pi}{2} - h\right)\right]$$

$$= \lim_{h\to 0} \frac{1}{\sqrt{3}}\tan^{-1}\left(\sqrt{3}\cot h\right)$$

$$= \lim_{h\to 0} \frac{1}{\sqrt{3}}\tan^{-1}(\to \infty) = \frac{\pi}{2\sqrt{3}} \qquad \text{.............(i)}$$

$$\underset{\text{at } x=\pi/2}{\text{RHL}} = \lim_{h\to 0} \frac{1}{\sqrt{3}}\tan^{-1}\sqrt{3}\left[\tan\left(\frac{\pi}{2} + h\right)\right]$$

$$= \lim_{h\to 0} \frac{1}{\sqrt{3}}\tan^{-1}\left(-\sqrt{3}\cot h\right)$$

$$= \lim_{h\to 0} \frac{1}{\sqrt{3}}\tan^{-1}(\to -\infty) = -\frac{\pi}{2\sqrt{3}} \qquad \text{.............(ii)}$$

From (i) and (ii), LHL $\neq$ RHL at $x = \pi/2$

$\Rightarrow$ Anti-derivative, F(x) is discontinuous at $x = \pi/2$

PART - B AREA UNDER CURVE

Example : 24

Find the area bounded by the curve $y = x^2 - 5x + 6$, X-axis and the lines $x = 1$ and $x = 4$.

Solution

For $y = 0$, we get $x^2 + 5x + 6 = 0$

$\Rightarrow \quad x = 2, 3$

Hence Area = $\int_1^2 y\,dx + \left|\int_2^3 y\,dx\right| + \int_3^4 y\,dx$

$$\Rightarrow \quad A = \int_1^2 (x^2 - 5x + 6)\,dx + \left|\int_2^3 (x^2 - 5x + 6)\,dx\right| + \int_5^4 (x^2 - 5x + 6)\,dx$$

$$\int_1^2 (x^2 - 5x + 6)\,dx = \frac{2^2 - 1^3}{3} - 5\left(\frac{2^2 - 1^2}{2}\right) + 6\,(2 - 1) = \frac{5}{6}$$

$$\int_2^3 (x^2 - 5x + 6)\,dx = \frac{3^3 - 2^3}{3} - 5\left(\frac{3^2 - 2^2}{2}\right) + 6\,(3 - 2) = -\frac{1}{6}$$

$$\int_3^4 (x^2 - 5x + 6)\,dx = \frac{4^3 - 3^3}{3} - 5\left(\frac{4^2 - 3^2}{2}\right) + 6\,(4 - 3) = \frac{5}{6}$$

$$\Rightarrow \quad A = \frac{5}{6} + \left|-\frac{1}{6}\right| + \frac{5}{6} = \frac{11}{6} \text{ sq. units.}$$

Example : 25

Find the area bounded by the curve : $y = \sqrt{4 - x}$, X-axis and Y-axis

Solution

Trace the curve $y = \sqrt{4 - x}$

1. Put $y = 0$ in the given curve to get $x = 4$ as the point of intersection with X-axis.
Put $x = 0$ in the given curve to get $y = 2$ as the point of intersection with Y-axis.

2. For the curve, $y = \sqrt{4 - x}$, $4 - x \geq 0$

$\Rightarrow \quad x \leq 4$

$\Rightarrow$ curve lies only to the left of $x = 4$ line.

3. As y is positive, curve is above X-axis.

Using steps 1 to 3, we can draw the rough sketch of $y = \sqrt{4 - x}$.

In figure

$$\text{Bounded area} = \int_0^4 \sqrt{4 - x}\;dx = \left|\frac{-2}{3}(4 - x)\sqrt{4 - x}\right|_0^4 = \frac{16}{3} \text{ sq. units.}$$

Example : 26

Find the area bounded by the curves $y = x^2$ and $x^2 + y^2 = 2$ above X-axis.

Solution

Let us first find the points of intersection of curves.

Solving $y = x^2$ and $x^2 + y^2 = 2$ simultaneously, we get :

$x^2 + x^4 = 2$

$\Rightarrow \quad (x^2 - 1)(x^2 + 2) = 0$

$\Rightarrow \quad x^2 = 1$ and $x^2 = -2$ (reject)

$\Rightarrow \quad x = \pm 1$

$\Rightarrow$ A ≡ (–1, 0) and B ≡ (1, 0)

Shaded Area = $\int_{-1}^{+1}\left(\sqrt{2-x^2}-x^2\right)dx$

$= \int_{-1}^{+1}\left(\sqrt{2-x^2}\,dx\right)-\int_{-1}^{+1}x^2\,dx$

$= 2\int_{0}^{1}\sqrt{2-x^2}\,dx-2\int_{0}^{1}x^2dx$

$= 2\left|\frac{x}{2}\sqrt{2-x^2}+\frac{2}{2}\sin^{-1}\frac{x}{\sqrt{2}}\right|_0^1-2\left(\frac{1}{3}\right)$

$= 2\left(\frac{1}{2}+\frac{\pi}{4}\right)-\frac{2}{3}$

$= \frac{1}{3}+\frac{\pi}{3}$ sq. units.

Example : 27

Find the area bounded by $y = x^2 - 4$ and $x + y = 2$

Solution

After drawing the figure, let us find the points of intersection of

$y = x^2 - 4$ and $x + y = 2$

$\Rightarrow$ $x + x^2 - 4 = 2$ $\Rightarrow$ $x^2 + x - 6 = 0$ $\Rightarrow$ $(x + 3)(x - 2) = 0$

$\Rightarrow$ $x = -3, 2$

$\Rightarrow$ A ≡ (–3, 0) and B ≡ (2, 0)

Shaded area $= \int_{-3}^{2}\left[(2-x)-(x^2-4)\right]dx$

$= \int_{-3}^{2}(2-x)\,dx-\int_{-3}^{2}(x^2-4)\,dx$

$= \left|2x-\frac{x^2}{2}\right|_{-3}^{2}-\left|\frac{x^3}{2}-4x\right|_{-3}^{2}$

$= 2 \times 5 - \frac{1}{2}(4-9)-\frac{1}{3}(8+27)+4(5)=\frac{125}{6}$

Example : 28

Find the area bounded by the circle $x^2 + y^2 = a^2$.

Solution

$x^2 + y^2 = a^2$ $\Rightarrow$ $y = \pm\sqrt{a^2-x^2}$

Equation of semicircle above X-axis is $y = +\sqrt{a^2-x^2}$

Area of circle = 4 (shaded area)

$= 4\int_{0}^{a}\sqrt{a^2-x^2}\,dx$

$$= 4 \left| \frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a} \right|_0^a = 4\ \frac{a^2}{2}\left(\frac{\pi}{3}\right) = \pi a^2$$

Example : 29

Find the area bounded by the curves $x^2 + y^2 = 4a^2$ and $y^2 = 3ax$.

Solution

The points of intersection A and B can be calculated by solving $x^2 + y^2 = 4a^2$ and $y^2 = 3ax$

$\Rightarrow \quad \left(\frac{y^2}{3a}\right)^2 = y^2 = 4a^2$

$\Rightarrow \quad y^4 + 9a^2 y^2 - 36a^4 = 0$

$\Rightarrow \quad (y^2 - 3a^2)(y^2 + 12a^2) = 0$

$\Rightarrow \quad y^2 = 3a^2$

$\Rightarrow \quad y^2 = -12a^2$ (reject)

$\Rightarrow \quad y^2 = 3a^2 \quad \Rightarrow \quad y = \pm\sqrt{3}a$

$\Rightarrow \quad y_A = \sqrt{3}a^2$ and $y_B = -\sqrt{3}a$

The equation of right half of $x^2 + y^2 = 4a^2$ is $x = \sqrt{4a^2 - y^2}$

$$\text{Shaded area} = \int_{-\sqrt{3}a}^{\sqrt{3}a} \left(\sqrt{4a^2 - y^2} - \frac{y^2}{3a}\right) dy$$

$$= 2\int_0^{\sqrt{3}a} \left(\sqrt{4a^2 - y^2} - \frac{y^2}{3a}\right) dy \qquad \text{(using property – 8)}$$

$$= 2\left| \frac{y}{2}\sqrt{4a^2 - y^2} + \frac{4a^2}{2}\sin^{-1}\frac{y}{2a} \right|_0^{\sqrt{3}a} - \frac{2}{3a}\left| \frac{y^3}{3} \right|_0^{\sqrt{3}a}$$

$$= \sqrt{3}a^2 + 4a^2\ \frac{\pi}{3} - \frac{2}{9a}\ 3\sqrt{3}\ a^3$$

$$= \left(\frac{1}{\sqrt{3}} + \frac{4\pi}{3}\right) a^2$$

Alternative Method :

shaded area = 2 × (area above X-axis)

x-coordinate of A = $\frac{y^2}{3a} = \frac{3a^2}{3a} = a$

The given curves are $y = \pm\sqrt{3ax}$ and $y = \pm\sqrt{4a^2 - x^2}$

But above the X-axis, the equations of the parabola and the circle are $\sqrt{3ax}$ and $y = \sqrt{4a^2 - x^2}$ respectively.

$$\Rightarrow \quad \text{shaded area} = 2\left[\int_0^a \sqrt{3ax}\,dx + \int_a^{2a} \sqrt{4a^2 - x^2}\,dx\right]$$

Solve it yourself to get the answer.

Example : 30

Find the area bounded by the curves : $y^2 = 4a(x + a)$ and $y^2 = 4b(b - x)$.

Solution

The two curves are :

$y^2 = 4a(x + a)$ ....(i)

and $y^2 = 4b(b - x)$ ....(ii)

Solving $y^2 = 4a(x + a)$ and $y^2 = 4b(b - x)$ simultaneously,

we get the coordinates of A and B.

Replacing values of x from (ii) into (i), we get :

$$y^2 = 4a\left(b - \frac{y^2}{4b} + a\right)$$

$\Rightarrow$ $y = \pm\sqrt{4ab}$ and $x = b - a$

$\Rightarrow$ $A \equiv -(b - a, \sqrt{4ab})$ and $B \equiv (b - a, -\sqrt{4ab})$

$$\text{shaded area} = \int_{-\sqrt{4ab}}^{\sqrt{4ab}} \left[\left(b - \frac{y^2}{4b}\right) - \left(\frac{y^2}{4b} - a\right)\right] dy$$

$\Rightarrow$ $A = 2(a + b)\sqrt{4ab} - \int_0^{\sqrt{4ab}} \left(\frac{y^2}{2b} + \frac{y^2}{2a}\right) dy$ (using property – 8)

$\Rightarrow$ $A = 2(a + b)\sqrt{4ab} - \frac{1}{2}\left[\frac{4ab\sqrt{4ab}}{3b} + \frac{4ab\sqrt{4ab}}{3a}\right]$

$\Rightarrow$ $A = 2(a + b)\sqrt{4ab} - \frac{2}{3}(a + b)\sqrt{4ab}$

$\Rightarrow$ $A = \frac{8}{3}(a + b)\sqrt{ab}$

Example : 31

Find the area bounded by the hyperbola : $x^2 - y^2 = a^2$ and the line $x = 2a$.

Solution

Shaded area = 2 × (Area of the portion above X-axis)

The equation of the curve above x-axis is : $y = \sqrt{x^2 - a^2}$

$\Rightarrow$ required area (A) $= 2\int_a^{2a} \sqrt{x^2 - a^2}\, dx$

$\Rightarrow$ $A = 2\left|\frac{x}{2}\sqrt{x^2 - a^2} - \frac{a^2}{2}\log\left|x + \sqrt{x^2 - a^2}\right|\right|_a^{2a}$

$\Rightarrow$ $A = 2\sqrt{3}\,a^2 - a^2 \log|(2a + \sqrt{3}a| + a^2 \log a$

$\Rightarrow$ $A = 2\sqrt{3}\,a^2 - a^2 \log(2 + \sqrt{3})$

Alternative Method :

$$\text{Area (A)} = \int_{yB}^{yA} \left(2a - \sqrt{a^2 + y^2}\right) dy$$

$\Rightarrow \quad A = \int_{-\sqrt{3}a}^{\sqrt{3}a} \left(2a - \sqrt{a^2 + y^2}\right) dy$

Example : 32

Find the area bounded by the curves : $x^2 + y^2 = 25$, $4y = |4 - x^2|$ and $x = 0$ in the first quadrant.

Solution

First of all find the coordinates of point of intersection. A by solving the equations of two gives curves :

$\Rightarrow \quad x^2 + y^2 = 25 \quad$ and $\quad 4y = |4 - x^2|$

$\Rightarrow \quad x^2 + \frac{(4 - x^2)^2}{16} = 25$

$\Rightarrow \quad (x^2 - 4)^2 + 16x^2 = 400$

$\Rightarrow \quad (x^2 + 4)^2 = 400$

$\Rightarrow \quad x^2 = 16$

$\Rightarrow \quad x = \pm 4$

$\Rightarrow \quad y = \frac{|4 - x^2|}{4} = 3$

$\Rightarrow$ Coordinates of point are $A \equiv (4, 3)$

Shaded are = $\int_0^4 \left[\sqrt{25 - x^2} - \frac{|4 - x^2|}{4}\right] dx$

$\Rightarrow \quad A = \int_0^4 \sqrt{25 - x^2}\, dx - \frac{1}{4}\int_0^4 |4 - x^2|\, dx$ ....(i)

Let $\quad I = \frac{1}{4} \int_0^4 |4 - x^2|\, dx$

$\Rightarrow \quad A = \frac{1}{4} \int_0^2 (4 - x^2)\, dx + \frac{1}{4}\int_2^4 (x^2 - 4)\, dx$

$\Rightarrow \quad A = \frac{1}{4} \left(\frac{64}{3} - 16\right) - \frac{1}{4} \left(\frac{8}{3} - 8\right) = 4$

On substituting the value of I in (i), we get :

$$A = \int_0^4 \sqrt{25 - x^2}\, dx - 4$$

$\Rightarrow \quad A = \left| \frac{x}{2}\sqrt{25 - x^2} + \frac{25}{2}\sin^{-1}\frac{x}{5} \right|_0^4 - 4$

$\Rightarrow \quad A = 6 + \frac{25}{2} \sin^{-1} \frac{4}{5} - 4 = 2 + \frac{25}{2} \sin^{-1} \frac{4}{5}$

Example : 33

Find the area enclosed by the loop in the curve : $4y^2 = 4ax^2 - x^3$.

Solution

The given curve is : $4y^2 = 4ax^2 - x^3$

To draw the rough sketch of the given curve, consider the following steps :

(1) On replacing y by –y, there is no change in function. It means the graph is symmetric about Y-axis

(2) For x = 4, y = 0 and for x = 0, y = 0

(3) In the given curve, LHS is positive for all values of y.

$\Rightarrow$ RHS ≥ 0 $\Rightarrow$ $x^2 (1 - x/4) \geq 0$ $\Rightarrow$ $x \leq 4$

Hence the curve lines to the left of x = 4

(4) As $x \to -\infty$, $y \to \pm \infty$

(5) Points of maximum/minimum :

$$8y \frac{dy}{dx} = 8x - 3x^2$$

$$\frac{dy}{dx} = 0 \quad \Rightarrow \quad x = 0, \frac{8}{3}$$

At x = 0, derivative is not defined

By checking for $\frac{d^2y}{dx^2}$, $x = \frac{8}{3}$ is a point of local maximum (above X-axis)

From graph

Shaded area (A) = 2 × (area of portion above X-axis)

$$\Rightarrow \quad A = 2\int_0^4 \frac{x}{2}\sqrt{4-x}\, dx = \int_0^4 x\sqrt{4-x}\ dx$$

$$\Rightarrow \quad A = \int_0^4 (4-x)\sqrt{4-(4-x)}\ dx \qquad \text{(using property – 4)}$$

$$\Rightarrow \quad A = \int_0^4 (4-x)\sqrt{x}\ dx$$

$$\Rightarrow \quad A = 4\left|\frac{2}{3}x\sqrt{x}\right|_0^4 - \left|\frac{2}{5}x^2\sqrt{x}\right|_0^4$$

$$\Rightarrow \quad A = \frac{128}{15} \text{ sq. units.}$$

Example : 34

Find the area bounded by the parabola $y = x^2$, X-axis and the tangent to the parabola at (1, 1)

Solution

The given curve is $y = a^2$

Equation of tangent at $A \equiv (1, 1)$ is :

$$y - 1\ y - 1 = \left.\frac{dy}{dx}\right]_{x=1} (x - 1) \qquad [\text{using : } y - y_1 = m(x - x_1)]$$

$\Rightarrow$ y – 1 = 2 (x – 1)

$\Rightarrow$ y = 2x – 1 ....(i)

The point of intersection of (i) with X-axis is B = (1/2, 0)

Shaded area = area (OACO) – area (ABC)

$\Rightarrow \quad \text{area} = \int_0^1 x^2\, dx - \int_{1/2}^1 (2x-1)\, dx$

$\Rightarrow \quad \text{area} = \frac{1}{3} - \left[1 - \frac{1}{4} - (1 - 1/2)\right]$

$\Rightarrow \quad \text{area} = \frac{1}{12}$

Example : 35

Evaluate : $\displaystyle\int_0^{\pi} \frac{x\sin(2x)\sin\left(\frac{\pi}{2}\cos x\right) dx}{2x - \pi}$

Solution

Let $\quad I = \displaystyle\int_0^{\pi} \frac{x\sin(2x)\sin\left(\frac{\pi}{2}\cos x\right) dx}{2x - \pi}$ ....(i)

Apply property – 4 to get

$\Rightarrow \quad I = \displaystyle\int_0^{\pi} \frac{(\pi - x)\sin(2\pi - 2x)\sin\left(\frac{\pi}{2}\cos(\pi - x)\right) dx}{2(\pi - x) - \pi}$

$= \displaystyle\int_0^{\pi} \frac{(\pi - x)\sin 2x \sin\left(\frac{\pi}{2}\cos x\right) dx}{2x - \pi}$ ....(ii)

Add (i) and (ii) to get

$2I = \displaystyle\int_0^{\pi} \sin 2x \sin\left[\frac{\pi}{2}\cos x\right] dx$

Let $\quad \frac{\pi}{2}\cos x = t \quad \Rightarrow \quad -\frac{\pi}{2}\sin x\, dx = dt$

$\Rightarrow \quad I = -\frac{4}{\pi^2}\displaystyle\int_{-\pi/2}^{\pi/2} t\sin t\, dt = \frac{8}{\pi^2}\int_0^{\pi/2} t\sin t\, dt$

$\Rightarrow \quad I = \frac{8}{\pi^2}\left[t\displaystyle\int_0^{\pi/2}\sin t\, dt + \int_0^{\pi/2}\cos t\, dt\right]$

$\Rightarrow \quad I = \frac{8}{\pi^2}\left[-\left| t\cos t\right|_0^{\pi/2} + (\sin t)_0^{\pi/2}\right] = \frac{8}{\pi^2}\,[0 + 1] = \frac{8}{\pi^2}$

Example : 36

Prove that : $\int_0^{\pi} \theta^3 \log\sin\theta d\theta = \frac{3\pi}{2}\int_0^{\pi} \theta^2 \log\left(\sqrt{2}\,\sin\theta\right)d\theta$

Solution

Let $I = \int_0^{\pi} \theta^3 \log\sin\theta\, d\theta$

Using property – 4, we get :

$$I = \int_0^{\pi} (\pi-\theta)^3 \log(\pi-\theta)\, d\theta = \int_0^{\pi} [\pi^3 - \theta^3 - 3\pi^2\theta + 3\pi\theta^2] \log\sin\theta\, d\theta$$

$$\Rightarrow \quad I = \pi^3 \int_0^{\pi} \log\sin\theta - \int_0^{\pi} \theta^3 \log\sin\theta d\theta - 3\pi\int_0^{\pi} \theta\log\sin\theta\, d\theta + 3\pi\int_0^{\pi} \theta^2 \log\sin\theta d\theta$$

$$\Rightarrow \quad 2I = \pi^3 \int_0^{\pi} \log\sin\theta d\theta = 3\pi^2 I_1 + 3\pi\int_0^{\pi} \theta^2 \log\sin\theta d\theta \quad ....(i)$$

Consider I_1 $\quad I_1 = \int_0^{\pi} \theta\log\sin\theta d\theta$

Using property – 4,

$$\text{we get } I_1 = \int_0^{\pi} (\pi-\theta)\log\sin\theta d\theta = \pi\int_0^{\pi} \log\sin\theta - \int_0^{\pi} \log\sin\theta$$

$$\Rightarrow \quad 2I_1 = \pi \int_0^{\pi} \log\sin\theta d\theta = 2\pi\int_0^{\pi/2} \log\sin\theta d\theta \qquad \text{[using property – 6]}$$

$$\Rightarrow \quad I_1 = -\frac{\pi^2}{2} \log 2 \qquad \text{using : } \int_0^{\pi/2} \log\sin\theta\, d\theta = \frac{-\pi}{2}\log 2$$

On Replacing value of I_1 in (i) we get,

$$2I = -\pi^4 \log 2 - 3\pi^2 \left(\frac{\pi^2}{2}\log 2\right) + 3\pi \int_0^{\pi} \theta^2 \log\sin\theta\, d\theta$$

$$= \frac{\pi^4}{2} \log 2 + 2\, 3\pi \int_0^{\pi} \theta^2 \log\sin\theta = 3\pi \int_0^{\pi} \left(\log\sqrt{2}\right)\theta^2 d\theta + 3\pi\int_0^{\pi} \theta^2 \log\sin\theta\, d\theta$$

$$= 3\pi \int_0^{\pi} \theta^2 \log\left(\sqrt{2}\sin\theta\right)d\theta$$

$$\Rightarrow \quad I = \frac{3}{2}\pi \int_0^{\pi} \theta^2 \log\sqrt{2}\sin\theta\, d\theta$$

Example : 37

Determine the value of $\int_{-\pi}^{\pi} \frac{2x(1+\sin x)}{1+\cos^2 x}$

Solution

$$I = \int_{-\pi}^{\pi} \frac{2x(1+\sin x)}{1+\cos^2 x}\,dx = 2\int_{-\pi}^{\pi} \frac{2x\sin x}{1+\cos^2 x} \qquad \left[\text{using}: \int_{-a}^{a} f(x)\,dx = \int_{0}^{a} f(x)+f(-x)\,dx\right]$$

$$\Rightarrow \quad I = 4\int_{0}^{\pi} \frac{x\sin x}{1+\cos^2 x}\,dx$$

$$\Rightarrow \quad 2I = 4\int_{0}^{\pi} \frac{\pi\sin x}{1+\cos^2 x}\,dx \qquad \left(\text{using} \int_{0}^{a} f(x)\,dx = \int_{0}^{a} f(a-x)\,dx\right)$$

$$\Rightarrow \quad I = 4\pi\int_{0}^{\pi/2} \frac{\sin x\,dx}{1+\cos^2 x}\,dx \qquad \left(\text{using} \int_{0}^{2a} f(x)\,dx = \int_{0}^{a} f(x)\,dx + \int_{0}^{a} f(2a-x)\,dx\right)$$

Put $\cos x = t$ $\Rightarrow$ $-\sin x\,dx = dt$

For $x = 0$, $t = 1$ and for $x = \pi/2$, $t = 0$

$$\Rightarrow \quad I = 4\pi\int_{0}^{1} \frac{dt}{1+t^2} = 4\pi \tan^{-1} t \Big|_{0}^{1} = 4\pi\,\frac{\pi}{4} = \pi^2$$

Example : 38

Let A_n be the area bounded by the curve $y = (\tan x)^n$ and the lines $x = 0$, $y = 0$ and $x = \pi/4$. Prove that for $n > 2$, $A_n + A_{n-2} = \frac{1}{n-1}$ and deduce $\frac{1}{2n+2} < An < \frac{1}{2n-2}$

Solution

According to the function, A_n is the area bounded by the curve $y = (\tan x)^n$, $x = 0$, $y = 0$ and $x = \pi/4$.

$$\text{So } A_n = \text{Shaded Area} = \int_{0}^{\pi/4} (\tan x)\,dx = \int_{0}^{\pi/4} \tan^2 x \tan^{n-2} x$$

$$\Rightarrow \quad A_n = \int_{0}^{\pi/4} (\sec^2 x - 1)\tan^{n-2} x = \int_{0}^{\pi/4} \sec^2 x \tan^{n-2} x - \int_{0}^{\pi/n} \sec^{n-2} x\,dx$$

$$\Rightarrow \quad A_n = \frac{\tan^{n-1} x}{n-1}\Bigg|_{0}^{\pi/4} - A_{n-2}$$

$$\Rightarrow \quad A_n + A_{n-2} = \frac{1}{n-1} \qquad \text{...........(i)}$$

Hence proved.

Replace n by n + 2 to get : $A_{n+2} + A_n = \frac{1}{n+1}$ ....(ii)

Observe that if n increases, $(\tan x)^n$ decreases because $0 \le \tan x \le 1$ $[0, \pi/4]$

$\Rightarrow$ As n is increased, A_n decreases.

$\Rightarrow$ $A_{n+2} < A_n < A_{n-2}$

Using (i) and (ii), replace values of A_{n-2} and A_{n+2} in terms of A_n to get,

$$\frac{1}{n+1} - A_n < A_n < \frac{1}{n-1} - A_n$$

$$\Rightarrow \quad \frac{1}{n+1} < 2A_n < \frac{1}{n-1} - A_n$$

$$\Rightarrow \quad \frac{1}{2n+1} < A_n < \frac{1}{2n-2}$$

Hence Proved.

Example : 39

Show that $\int_0^{n\pi+v} |\sin x|\, dx = 2n + 1 - \cos v$, where n is a +ve integer and $0 \le v \le \pi$

Solution

Let $I = \int_0^{n\pi+v} |\sin x| = \int_0^{n\pi} |\sin x| + \int_{n\pi}^{n\pi+v} |\sin x|$ (using property – 1)

$\Rightarrow \quad I = I_1 + I_2$ ....(i)

Consider I_1

$I_1 = \int_0^{n\pi} |\sin x| = n\int_0^{\pi} |\sin x|$ (using property – 9 and period of |sin x| is π)

$\Rightarrow \quad I_1 = n \int_0^{\pi} \sin x\, dx$ (As $\sin x \ge 0$ in $[0, \pi]$, $|\sin x| = \sin x$)

$\Rightarrow \quad I_1 = -n\, |\cos x|_0^{\pi} = -n\,[-1 -1] = 2n$

Consider I_2

$I_2 = \int_{n\pi}^{n\pi+v} |\sin x|\, dx$

Put $x = n\pi + \theta \quad \Rightarrow \quad dx = d\theta$

when x is $n\pi$, $\theta = 0$ and when $x = n\pi + v$, $\theta = v$

$\Rightarrow \quad I_2 = \int_0^{v} |\sin(\pi x + \theta)|\, d\theta = \int_0^{v} |\sin\theta|\, d\theta$ ($\because$ period of $|\sin x| = \pi$)

$\Rightarrow \quad I_2 = \int_0^{v} |\sin\theta|\, d\theta = \int_0^{v} \sin\theta\, d\theta$ ($\because$ for $0 \le \theta \le \pi$, $\sin\theta$ is positive)

$= -\, |\cos\theta|_0^{v} = 1 - \cos v$

On substituting the values of I_1 and I_2 in (i), we get

$I = 2n + 1\,(1 - \cos v) = 2n + 1 - \cos v$

Hence proved.

Example : 40

It is known that f(x) is an odd function in the interval $\left[-\frac{T}{2}, \frac{T}{2}\right]$ and has a period equal to T. Prove that

$\int_a^x f(t)\,dt$ is also periodic function with the same period.

Solution

It is given that : f(x) = – f(x) ....(i)

and f(x + t) = f(x) ....(ii)

Let $g(x) = \int_a^x f(t)\,dt$

$\Rightarrow \quad g(x+T) = \int_a^{x+T} f(t)\,dt = \int_a^x f(t)\,dt + \int_x^{T/2} f(t)\,dt + \int_{T/2}^{x+T} f(t)\,dt$ (using property – 1)

Put t y = y + T in the third integral on RHS.

$\Rightarrow$ dt = dy

when t = T/2, y = –T/2 and hwn t = x + T, y = x

$\Rightarrow \quad g(x+T) = \int_a^x f(t)\,dt + \int_x^{T/2} f(t)\,dt + \int_{-T/2}^{x} f(y+T)\,dy$

Using (i), we get $g(x+T) = \int_a^x f(t)\,dt + \int_x^{T/2} f(t)\,dt + \int_{-T/2}^{x} f(y)\,dy$

$g(x+T) = \int_a^x f(t)\,dt + \int_{-T/2}^{T/2} f(t)\,dt$ (using property – 1)

$\Rightarrow \quad g(x+T) = \int_a^x f(t)\,dt$ (using property – 8)

$\Rightarrow$ g(x + T) = g(x)

$\Rightarrow$ g(x) is also periodic function when period T.

Example : 41

Evaluate the integral $\int_{-1\sqrt{3}}^{1/\sqrt{3}} \frac{x^4}{1-x^4}\cos^{-1}\frac{2x}{1+x^2}\,dx$

Solution

$I = \int_{-1\sqrt{3}}^{1/\sqrt{3}} \frac{x^4}{1-x^4}\cos^{-1}\frac{2x}{1+x^2}\,dx = \int_{-1\sqrt{3}}^{1/\sqrt{3}} \frac{x^4}{1-x^4}\left[\frac{\pi}{2} - \sin^{-1}\frac{2x}{1+x^2}\right]dx$ (using : $\sin^{-1}x + \cos^{-1}x = \pi/2$)

$\Rightarrow \quad I = \frac{\pi}{2}\int_{-1\sqrt{3}}^{1/\sqrt{3}} \frac{x^4}{1-x^4}\,dx - \int_{-1\sqrt{3}}^{1/\sqrt{3}} \frac{x^4}{1-x^4}\sin^{-1}\frac{2x}{1+x^2}\,dx$

As integrand of second integral is an odd function, integral will be zero i.e.

$\Rightarrow \quad I = \frac{\pi}{2}\int_{-1\sqrt{3}}^{1/\sqrt{3}} \frac{x^4}{1-x^4}\,dx - 0$ [using property – 8]

$$= -\frac{2\pi}{2}\int_0^{1/\sqrt{3}} \frac{x^4-1+1}{x^4-1} = -\pi\int_0^{1/\sqrt{3}}\left(1+\frac{1}{x^4-1}\right)dx$$

$$\Rightarrow \qquad I = \frac{-\pi}{\sqrt{3}} + \frac{(-\pi)}{2}\int_0^{1/\sqrt{3}} \frac{x^2+1-(x^2-1)}{(x^2+1)(x^2-1)}\,dx$$

$$= -\frac{-\pi}{\sqrt{3}} - \frac{\pi}{2}\left[\int_0^{1/\sqrt{3}} \frac{1}{x^2-1} - \int_0^{1/\sqrt{3}} \frac{1}{x^2+1}\,dx\right]$$

$$= -\frac{-\pi}{\sqrt{3}} - \frac{\pi}{2}\left[\frac{1}{2}\left|\log\frac{x-1}{x+1}\right|_0^{1/\sqrt{3}} - \left|\tan^{-1}x\right|_0^{1/\sqrt{3}}\right]$$

$$= -\frac{\pi}{\sqrt{3}} + \frac{\pi^2}{12} - \frac{\pi}{4}\log\frac{\sqrt{3}-1}{\sqrt{3}+1}$$

Example : 42

If f is a continuous function $\int_0^x f(t)\,dt \to \infty$, then show that every line y = mx intersect the curve

$y^2 + \int_0^x f(t)\,dt = 2$

Solution

If y = mx and $y^2 + \int_0^X f(t)\,dt = 2$ have to intersect for all value of m, then

$m^2x^2 + \int_0^X f(t)\,dt = 2$ must posses atleast one solution (root) for all m. ....(i)

Let $g(x) = m^2x^2 + \int_0^x f(t)\,dt - 2$

For (i) to e true, g(x) should be zero for atleast one value of x.
As f(x) is a given continuous function and m^2x^2 is a continuous function,

$g(x) = m^2x^2 + \int_0^x f(t)\,dt$ is also a continuous function ....(iii)

($\because$ because sum of two continuous functions is also continuous)

g(0) = – 2 and $\lim_{x\to\infty} g(x) = \infty$ ....(iii)

Combining (ii) and (iii), we can say that :
for all values of m, the curve g(x), intersect the y = 0 line (i.e. X-axis) for atleast one value of x.
$\Rightarrow$ g(x) = 0 has atleast one solution for all values of m.
Hence proved

Example : 43

Let a + b = 4, where a < 2, and let g(x) be a differentiable function. If $\frac{dg}{dx}$ > 0 for all x, prove that

$\int_0^a g(x)\,dx + \int_0^b g(x)\,dx$ increases as (b – a) increases

Solution

Let $b - a = t$ ....(i)

It si given that $a + b = 4$ ....(ii)

Solving (i) and (ii), we get $b = \frac{t+4}{2}$ and $a = \frac{4-t}{2}$

As a < 2, $\frac{4-t}{2} < 2$

$\Rightarrow \quad 4 - t < 4 \quad \Rightarrow \quad t > 0$

Let $f(t) = \int_0^a g(x)\,dx + \int_0^b g(x)\,dx$

$\Rightarrow \quad f(t) = \int_0^{\frac{4-t}{2}} g(x)\,dx + \int_0^{\frac{t+4}{2}} g(x)\,dx$

$f'(t) = g\left(\frac{4-t}{2}\right)\left(-\frac{1}{2}\right) + g\left(\frac{4+t}{2}\right)\left(\frac{1}{2}\right) = \frac{1}{2}\left[g\left(\frac{4+t}{2}\right) - g\left(\frac{4-t}{2}\right)\right]$ ....(i)

As $\frac{dg}{dx}$. 0, g(x) is an increasing function.

For t > 0, $\frac{4+t}{2} > \frac{4-t}{2}$

$\Rightarrow \quad g\left(\frac{4+t}{2}\right) > g\left(\frac{4-t}{2}\right)$ [$\because$ g(x) is an increasing function]

$\Rightarrow \quad f'(t) > 0 \ \forall\ t > 0$ [using (i)]

$\Rightarrow$ f(t) is an increasing function as t increases.

Hence Proved.

Example : 44

Find the area between the curve $y = 2x^4 - x^2$, the x-axis and the ordinates of two minima of the curve.

Solution

Using the curve tracing steps, draw the rough sketch of the function $y = 2x^4 - x^2$.

Following are the properties of the curve which can be used to draw its rough sketch

(i) The curve is symmetrical about y-axis

(ii) Point of intersection with x-axis are x = 0, $x = \pm 1/\sqrt{2}$. Only point of intersection with y-axis is y = 0.

(iii) For $x \in \left(-\infty - \frac{1}{\sqrt{2}}\right) \cup \left(\frac{1}{\sqrt{2}}\ \infty\right)$, y > 0 i.e. curve lies above x-axis and in the other intervals it lies below x-axis.

(iv) Put $\frac{dy}{dx}$ = 0 to get $x = \pm 1/2$ as the points of local minimum. On plotting the above information on graph, we get the rough sketch of the graph. The shaded area in the graph is the required area

Required Area = $2\left|\int_0^{1/2}(2x^4 - x^2)\,dx\right| = 2\left|\left[\frac{2x^5}{5} - \frac{x^3}{3}\right]_0^{1/2}\right| = \frac{7}{120}$

Example : 45

Consider a square with vertices at (1, 1), (–1, –1), (–1, 1) and (1, –1). Let S be the region consisting of all points inside the square which are nearer to the origin than to the edge. Sketch the region S and find its area.

Solution

Let ABCD be the square with vertices A(1, 1), B(–1, 1), C(1, –1) and D(1, –1). The origin O is the centre of this square. Let (x, y) be a moving point in the required region. Then :

$$\sqrt{x^2+y^2} < |1 - x|,\ \sqrt{x^2+y^2} < |1 + x|,\ \sqrt{x^2+y^2} < |1 - y|,\ \sqrt{x^2+y^2} < |1 + y|$$

i.e. $x^2 + y^2 < (1 - x)^2$, $x^2 + y^2 < (1 + x)^2$, $x^2 + y^2 < (1 - y)^2$, $x^2 + y^2 < (1 + y)^2$

$\Rightarrow$ $y^2 = 1 - 2x$ ....(i)

$y^2 = 1 + 2x$ ....(ii)

$x^2 = 1 - 2y$ ....(iii)

$x^2 = 1 + 2y$ ....(iv)

Plotting the curves (i) to (iv), we can identify that the area bounded by the curves is the shaded area (i.e. region lying inside the four curves).

Required Area = 4 × Area (OPQR) = 4 [Area (OSQRO) + Area (SPQS)]

= 4 [Area (OSQRO) + Area (SPQS)]

$$= 4\left[\int_0^{x_s}\frac{1}{2}(1-x^2)\,dx + \int_{x_s}^{1/2}\sqrt{1-2x}\,dx\right] \quad (x_s \text{ is the x-coordinate of point S}) \quad ....(v)$$

To find x_s, solve curves (i) and (iii)

$\Rightarrow$ $x^2 - y^2 = -2(y - x)$

$\Rightarrow$ $(x - y)[x + y - 2] = 0$ $\Rightarrow$ $x = y$

Replace x = y in (i) to get $x^2 + 2x - 1 = 0$ $\Rightarrow$ $x = \sqrt{2} \pm 1$

(Check yourself that for x + y = 2, these is no point of intersection between the lines)

As x < 1, S is ($\sqrt{2} - 1$, $\sqrt{2} - 1$)

replacing the value of x_s in (i), we get

$$\text{Required Area} = 4\left[\int_0^{\sqrt{2}-1}\frac{1}{2}(1-x^2) + \int_{\sqrt{2}-1}^{1/2}\sqrt{1-2x}\,dx\right]$$

$$= 4\left[\frac{1}{2}\left(x - \frac{x^3}{3}\right)\right]_0^{\sqrt{2}-1} - \frac{2}{3}\times\frac{1}{2}(1-2x)^{3/2}\Big]_{\sqrt{2}-1}^{1/2} = \frac{2}{3}\left(8\sqrt{2} - 10\right) \text{ sq. units}$$

Example : 46

Let O(0, 0), A(2, 0) and B (1, 1/$\sqrt{3}$) be the vertices of a triangle. Let R be the region consisting of all those points P inside ΔOAB which satisfy d(P, OA) $\le$ min {d (P, AB)}, where d denotes the distance from the point to the corresponding line. Sketch the region R and find its area.

Solution

Let the coordinates of moving point P be (x, y)

Equation of line OA $\equiv$ y = 0

Equation of line OB $\equiv$ $\sqrt{3}$ = x

Equation of line AB $\equiv$ $\sqrt{3}$y = 2 – x.

d(P, OA) = distance of moving point P from line OA = y

d(P, OB) = distance of moving point P from line OB = $\frac{|\sqrt{3}y - x|}{2}$

d(p, AB) = distance of moving point P from line AB = $\frac{|\sqrt{3}y + x - 2|}{2}$

It is given in the question that P moves inside the triangle OAB according to the following equation.
d (P, OA) ≤ min {d(P, OB), d (P, AB)}

⇒ $y \le \min\left\{\frac{|\sqrt{3}y - x|}{2}, \frac{|\sqrt{3}y + x - 2|}{2}\right\}$

⇒ $y \le \frac{|\sqrt{3}y - x|}{2}$ ....(i) and $y \le \frac{|\sqrt{3}y + x - 2|}{2}$ ....(ii)

Consider (i) $y \le \frac{|\sqrt{3}y - x|}{2}$

$y \le \frac{x - \sqrt{3}y}{2}$ i.e. $x > \sqrt{3}y$ because P(x, y) moves inside the triangle, below the lines OB

⇒ $(2 + \sqrt{3})\, y \le x$
⇒ $y \le (2 - \sqrt{3})\, x$
⇒ $y \le \tan 15^o\, x$. (Note : $y = \tan 15^o x$ is an acute ∠ bisector of ∠ O] ....(iii)

Consider (ii) $y \le \frac{|\sqrt{3}y + x - 2|}{2}$

⇒ $2y \le 2 - x - \sqrt{3}y$
i.e. $\sqrt{3}y + x - 2 < 0$ because P(x, y) moves inside the triangle, below the line AB.
⇒ $(2 + \sqrt{3})\, y \le -(x - 2)$
⇒ $y \le -(2 - \sqrt{3})(x - 2)$
⇒ $y \le -\tan 15^o (x - 2)$ [Note : $y = -\tan 15^o (x - 2)$ is an acute ∠ bisector of ∠A]
From (iii) and (iv), P moves inside the triangle as shown in figure. (shaded area).
Let D be the foot of the perpendicular from C to OA
As ∠COA = ∠OAC = 15º, ΔOCA is an isosceles Δ .
⇒ OD = AD = 1 unit.

Area of shaded region = Area of ΔOCA = 1/2 base × height = $\frac{1}{2}$ (2) [1 tan 15º] = tan 15º = $2 - \sqrt{3}$

Alternate Method
Let acute angle bisector fo angles O and A meet at point C inside the triangle ABC.
Consider OC
On Line OC, d(P, OA) = d(p, OB) [note if P moves on OC d(P, OB) < d (P, AB)]
⇒ Below the line OC, d(P, OA) < d(p, OB) < d (P, AB) ....(i)
On Line AC, d(P, OA) = d(P, AB) [note if P moves on AC d(P, AB) < d (P, OB)]
⇒ Below the line OC, d(P, OA) > d (P, AB) < d (P, OB) ....(ii)
On combining (i) and (ii), P moves inside the triangle OAC
Now the required area is the area of the triangle OAC = $2 - \sqrt{3}$ (refer previous method)

Example : 47

Sketch the smaller of the regions bonded by the curves $x^2 + 4y^2 - 2x - 8y + 1 = 0$ and $4y^2 - 3x - 8y + 7 = 0$. Also find its area.

Solution

Express the two curves in perfect square form to get :

$\frac{(x-1)^2}{4} + (y - 1)^2 = 1$ ....(i)

[i.e. ellipse centred at (1, 1)]

and $(y-1)^2 = \frac{3}{4}(x-1)$ ....(ii)

[i.e. parabola whose vertex is at (1, 1)]

To calculate the area bounded between curves (i) and (ii), it is convenient to shift the origin at (1, 1).
Replace x – 1 = X and y – 1 = Y in (i) and (ii).
The new equations of parabola and ellipse with shifted origin are :

$$\frac{X^2}{4} + Y^2 = 1 \quad ....(iii)$$

$$Y^2 = \frac{3}{4}X \quad ....(iv)$$

It can be easily observed that the area of the smaller region bounded by (i) and (ii) is the same as the area of the smaller region bounded by (iii) and (iv) on the X-Y plane i.e. Area bounded remains same in the two cases.
So area of region bounded by (iii) and (iv)

= shaded area shown in the figure

= 2 × shaded area lying in I^{st} quadrant

$$= 2\left[\int_0^{x_A} \frac{\sqrt{3}}{2}\sqrt{X}\,dX + \frac{1}{2}\int_{x_A}^{2}\sqrt{4-x^2}\,dX\right] \quad ....(v)$$

Solve curves (iii) and (iv) to get point of intersection $A = \left(1, \frac{\sqrt{3}}{2}\right)$

$\Rightarrow \quad x_A = 1$

Replace x_A in (v) to get :

$$\text{Required Area} = 2\left[\int_0^{1} \frac{\sqrt{3}}{2}\sqrt{X}\,dX + \frac{1}{2}\int_{1}^{2}\sqrt{4-X^2}\,dx\right]$$

$$= \frac{2}{\sqrt{3}}X^{3/2}\Big|_0^1 + \left[\frac{X}{2}\sqrt{4-X^2} + 2\sin^{-1}\frac{X}{2}\right]_1^2 = \frac{\sqrt{3}}{6} + \frac{2\pi}{3}$$