Example : 1

Solve the differential equation : $\frac{dy}{dx} = x$.

Solution

The given differential equation is : dy – xdx

$\Rightarrow \quad \int dy = \int xdx$

$\Rightarrow \quad y = \frac{x^2}{2} + C \quad$ ....(i)

where C is an arbitrary constant.

Note that (i) is the general solution of the given differential equation.

Example : 2

Solve the differential equation : $\frac{dy}{dx} = x - 1$ if y = 0 for x = 1.

Solution

The given differential equation is : dy = (x – 1) dx

$\int dy = \int (x-1)\,dx \quad \Rightarrow \quad y = \frac{x^2}{2} - x + C \quad$ (general solution)

This is the general solution. We can find value of C using y = 0 for x = 1.

$0 = \frac{1}{2} - 1 + C \quad \Rightarrow \quad C = \frac{1}{2}$

$y = \frac{x^2}{2} - x + \frac{1}{2}$ is the particular solution.

Example : 3

Solve the differential equation : (1 + x) y dx + (1 – y) x dy = 0

Solution

Separate the term of x and y to get : (1 + x) y dx = – (1 – y) x dy

$\Rightarrow \quad \frac{1+x}{x}\,dx = \frac{y-1}{y}\,dy$

$\Rightarrow \quad \int \left(\frac{1+x}{x}\right) dx = \int \left(\frac{y-1}{y}\right) dy$

$\Rightarrow \quad \log x + x = y - \log y + C$

$\Rightarrow \quad \log xy + x - y = C$ is the general solution.

Example : 4

Solve the differential equation : $xy^2 \frac{dy}{dx} = 1 - x^2 + y^2 - x^2y^2$

Solution

The given differential equation : $xy^2 \frac{dy}{dx} = 1 - x^2 + y^2 - x^2y^2$

$\Rightarrow \quad xy^2 \frac{dy}{dx} = (1 - x^2)(1 + y^2)$

$\Rightarrow \quad \frac{y^2 dy}{1+y^2} = \frac{(1-x^2)\,dx}{x}$

$\Rightarrow \quad \int \frac{y^2}{1+y^2}\,dy = \int\left(\frac{1}{x} - x\right) dx$

$\Rightarrow \quad y - \tan^{-1} y = \log x - \frac{x^2}{2} + C$ is the general solution of the given differential equation.

Example : 5

Solve $\frac{d^2y}{dx^2} = x + \sin x$ if $y = 0$ and $\frac{dy}{dx} = -1$ for $x = 0$

Solution

The given differential equation is : $\frac{d^2y}{dx^2} = x + \sin x$ ....(i)

It is an order 2 differential equation. But it can be easily reduced to order 1 differential equation by integrating both sides. On Integrating both sides of equation (i), we get

$$\frac{dy}{dx} = \int (x + \sin x)\,dx$$

$\Rightarrow \quad \frac{dy}{dx} = \frac{x^2}{2} - \cos x + C_1$, where C_1 is an arbitrary constant ....(ii)

$\Rightarrow \quad dy = (x^2/2 - \cos x + C_1)\,dx$

$\Rightarrow \quad \int dy = \int\left(\frac{x^2}{2} - \cos x + C_1\right) dx$

$\Rightarrow \quad y = \frac{x^3}{6} - \sin x + C_1x + C_2$

This is the genral solution. For particular solution, we have to find C_1 and C_2

for $x = 0, y = 0$ $\Rightarrow$ $0 = \frac{0^3}{6} - \sin 0 + 0\,C_1 + C_2$ $\Rightarrow$ $C_2 = 0$

for $x = 0$, $\frac{dy}{dx} = -1$ $\Rightarrow$ $-1 = \frac{0^2}{2} - \cos 0 + C_1$ $\Rightarrow$ $C_1 = 0$ [put $x = 0$ and $dy/dx = -1$ in (2)]

$\Rightarrow \quad y = \frac{x^3}{6} - \sin x$ is the particular solution of the given differential equation.

Example : 6

Solve the differential equation : $\frac{dy}{dx} - x \tan (y - x) = 1$

Solution

The given differential equation is : $\frac{dy}{dx} - x \tan (y - x) = 1$

Put $\quad z = y - x$

$\Rightarrow \quad \frac{dz}{dx} = \frac{dy}{dx} - 1 \quad \Rightarrow \quad \frac{dy}{dx} = \frac{dz}{dx} + 1$

$\Rightarrow$ the given equation becomes : $\left(\frac{dz}{dx} + 1\right) - x \tan z = 1$

$\Rightarrow \quad \frac{dz}{dx} = x \tan z$

$\Rightarrow \quad \int \cot z \, dz = \int x \, dx$

$\Rightarrow \quad \log \sin z = \frac{x^2}{2} + C$

$\Rightarrow \quad \sin(y - x) = e^{x^2/2} \,.\, e^C$

$\Rightarrow \quad \sin(y - x) = ke^{x^2/2}$ where k is an arbitrary constant.

Example : 7

Solve the differential equation : $\frac{dy}{dx} = \frac{2x - y}{x + y}$

Solution

The given differential equation is : $\frac{dy}{dx} = \frac{2x - y}{x + y}$

$\Rightarrow \quad \frac{dy}{dx} = \frac{2 - y/x}{1 + y/x}$

Let $y = mx$ $\quad \Rightarrow \quad \frac{dy}{dx} = m + x\frac{dm}{dx}$

$\Rightarrow \quad m + x\frac{dm}{dx} = \frac{2 - m}{1 + m}$

$\Rightarrow \quad x\frac{dm}{dx} = \frac{2 - 2m - m^2}{1 + m}$

$\Rightarrow \quad \frac{(1 + m)\,dm}{2 - 2m - m^2} = \frac{dx}{x}$

Integrate both sides :

$\Rightarrow \quad \frac{-1}{2} \int \frac{-2 - 2m}{2 - 2m - m^2} \, dm = \int \frac{dx}{x}$

$\Rightarrow \quad \frac{-1}{2} \log(2 - 2m - m^2) = \log x + \log C,$ where C is an arbitrary constant

$\Rightarrow \quad (2 - 2m - m^2) = \frac{1}{C^2 x^2}$

$\Rightarrow \quad \left(2 - \frac{2y}{x} - \frac{y^2}{x^2}\right) x^2 = K,$ where K is an arbitrary constant.

$\Rightarrow \quad 2x^2 - 2xy - y^2 = K$ is the required general solution.

Example : 8

Solve the differential equation : $xdy - ydx = \sqrt{x^2 + y^2}\ dx$

Solution

The given differential equation is : $xdy - ydx = \sqrt{x^2 + y^2}\ dx$

$$\Rightarrow \qquad \frac{dy}{dx} = \frac{y + \sqrt{x^2 + y^2}}{x}$$

Let $y = mx \qquad \Rightarrow \qquad \frac{dy}{dx} = m + x\,\frac{dm}{dx}$

$$\Rightarrow \qquad \frac{dm}{\sqrt{1+m^2}} = \frac{dx}{x}$$

$$\Rightarrow \qquad \int \frac{dm}{\sqrt{1+m^2}} = \int \frac{dx}{x}$$

$$\Rightarrow \qquad \log\left| m + \sqrt{1+m^2} \right| = \log x + \log C, \qquad \text{where C is an arbitrary constant.}$$

$$\Rightarrow \qquad \frac{y}{x} + \sqrt{1 + \frac{y^2}{x^2}} = Cx$$

Example : 9

Solve the differential equation : $(2x + y - 3)\ dy = (x + 2y - 3)\ dx$

Solution

The given differential equation is : $\frac{dy}{dx} = \frac{x - 2y - 3}{2x + y - 3}$

Solving $\left\{\begin{matrix} x + 2y - 3 = 0 \\ 2x + y - 3 = 0 \end{matrix}\right\}$, we get : $x = 1$, $y = 1$

Put $\quad x = u + 1 \qquad$ and $\qquad y = v + 1$

$$\Rightarrow \qquad \frac{dy}{dx} = \frac{dv}{du}$$

$$\Rightarrow \qquad \frac{dv}{du} = \frac{(1+u) + 2(1+v) - 3}{2(1+u) + (1+v) - 3} = \frac{u + 2v}{2u + v}$$

Now put $v = mu \qquad \Rightarrow \qquad \frac{dv}{du} = m + u\,\frac{dm}{du}$

$$\Rightarrow \qquad m + u\,\frac{dm}{du} = \frac{1 + 2m}{2 + m}$$

$$\Rightarrow \qquad \frac{2+m}{1-m^2}\,dm = \frac{du}{u}$$

$$\Rightarrow \qquad \int \frac{2+m}{1-m^2}\,dm = \int \frac{du}{u}$$

$$\Rightarrow \qquad \int \left\{ \frac{1/2}{1+m} + \frac{3/2}{1-m} \right\} dm = \int \frac{du}{u} \qquad \text{(Resolving into partial fractions)}$$

$\Rightarrow \quad \frac{1}{2}\log|1+m| - \frac{3}{2}\log|1-m| = \log u + \log C$

$\Rightarrow \quad (1+m)(1-m)^{-3} = u^2 C^2$ where $m = \frac{v}{u} = \frac{y-1}{x-1}$ and $u = x - 1$

$\Rightarrow \quad \left[1+\frac{y-1}{x-1}\right]\left[1-\frac{y-1}{x-1}\right]^{-3} = (x-1)^2 C^2$

$\Rightarrow \quad (x+y-2) = (x-y)^3 C^2$ where c^2 is a constant

Example : 10

Solve the differential equation : $x\frac{dy}{dx} + y = x^3$.

Solution

The given equation is : $x\frac{dy}{dx} + y = x^3$.

Convert to standard from by dividing by x.

$\Rightarrow \quad \frac{dy}{dx} + \frac{1}{x}y = x^2$

$\Rightarrow \quad P = \frac{1}{x}$ and $Q = x^2$

If $= e^{\int P\,dx} = e^{\int \frac{dx}{x}} = e^{\ln x} = x$

$\Rightarrow$ Solution is : $yx = \int x^2(x)\,dx + C$ (using the formula)

$\Rightarrow \quad xy = \frac{x^4}{4} = C$ is the genral solution

Example : 11

Solve $\sin x\frac{dy}{dx} + y\cos x = 2\sin^2 x\cos x$

Solution

The given differential equation is :

$\frac{dy}{dx} + \cot x\, y = 2\sin x\cos x$

$\Rightarrow \quad P = \cot x$ and $Q = 2\sin x\cos x$

$\int P\,dx = \int \cot x\,dx = \log\sin x$

$\Rightarrow \quad$ I.F. $= e^{\log\sin x} = \sin x$

Using the standard result, the solution is : y (I.F.) = $\int Q\,(\text{I.F.})\,dx + C$

$\Rightarrow \quad y\sin x = \int 2\sin x\cos x\sin x\,dx + C$

$\Rightarrow \quad y\sin x = \frac{2}{3}\sin^3 x + C$ is the general solution.

Example : 12

Solve the differential equation : $x^2 \dfrac{dy}{dx} + xy = y^2$.

Solution

The differential equation is : $\dfrac{dy}{dx} + \dfrac{y}{x} = \dfrac{y^2}{x^2}$ (Bernoulli's Differential Equation)

$$\Rightarrow \quad \frac{1}{y^2}\frac{dy}{dx} + \frac{1}{xy} = \frac{1}{x^2} \quad \text{.........(i)}$$

Let $\dfrac{1}{y} = t \Rightarrow \dfrac{-1}{y^2}\dfrac{dy}{dx} = \dfrac{dt}{dx}$

On substituting in (i), we get

$\dfrac{dy}{dx} - \dfrac{t}{x} = \dfrac{-1}{x^2}$ i.e. linear differential equation.

I.F. $= e^{\int -\frac{1}{x}dx} = e^{-lux} = \dfrac{1}{x}$

Using the standard result, the solution of the differential equation is :

$$\frac{t}{x} = -\int\left(\frac{1}{x}\right)\frac{1}{x^2}\,dx + C$$

$$\Rightarrow \quad \frac{1}{xy} = +\frac{1}{2x^2} + C \text{ is the general solution.}$$

Example : 13

Solve the differential equation : $y^2 \dfrac{dy}{dx} = x + y^3$.

Solution

The given differential equation is : $y^2 \dfrac{dy}{dx} = x + y^3$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{x}{y^2} + y$$

$$\Rightarrow \quad \frac{dy}{dx} - y = xy^{-2} \qquad \text{(Bernoulli's Differential Equation)}$$

$$\Rightarrow \quad y^2\frac{dy}{dx} - y^3 = x$$

Let $y^3 = t \Rightarrow 3y^2\dfrac{dy}{dx} = \dfrac{dt}{dx}$

On substituting in the differential equation, it reduces to linear differential equation : i.e.

$\dfrac{dt}{dx} - dt = 3x$

I.F. $= e^{\int -3\,dx} = e^{-3x}$

Using the standard result, the solution of the differential equation is :

$e^{-3x}\,t = 3\int xe^{-3x}\,dx + C$

$\Rightarrow \quad y^3 e^{-3x} = 3\left[x\int e^{-3x}\,dx + \frac{1}{3}\int e^{-3x}\,dx\right] + C$

$\Rightarrow \quad y^3 = -x - 1/3 + Ce^{3x}$

$\Rightarrow \quad 3(y^3 + x) + 1 = ke^{3x}$ is the general solution

Example : 14

Solve the differential equation : $xyp^2 - (x^2 - y^2)p - xy = 0$, where $\frac{dy}{dx} = p$.

Solution

The given differential equation is : $xyp^2 - x^2 p + y^2 p - xy = 0$

$\Rightarrow \quad (xyp^2 + y^2p) - (x^2p + xy) = 0$

$\Rightarrow \quad yp(xp + y) - x(xp + y) = 0$

$\Rightarrow \quad (xp + y)(yp - x) = 0$

Case – I $\quad x\frac{dy}{dx} + y = 0$

$\Rightarrow \quad xdy + ydx = 0 \quad \Rightarrow \quad d(xy) = 0$

On integrating, we get : $xy = k$

Case – II $\quad xp - x = 0$

$$y\frac{dy}{dx} - x = 0$$

integrating, we get $\frac{y^2}{2} - \frac{x^2}{2} = k$

or $\quad y^2 - x^2 - 2k = 0$

Hence the solution is $(xy - k)(y^2 - x^2 - 2k) = 0$

Example : 15

Solve the differential equation : $p(p + x) = y(x + y)$, where $p = \frac{dy}{dx}$

Solution

The given differential equation is : $p^2 + px - xy - y^2 = 0$

$\Rightarrow \quad (p^2 - y^2) + (px - xy) = 0$

$\Rightarrow \quad (p - y)(p + y) + x(p - y) = 0$

$\Rightarrow \quad (p - y)(p + x + y) = 0$

Case – I

$\Rightarrow \quad \frac{dy}{dx} - y = 0 \quad \Rightarrow \quad \frac{dy}{y} - dx = 0$

Integrating, we get : $\quad \log y = x + \log A = \log(Ae^x)$

or $\quad y = Ae^x$, where A is an arbitrary constant ....(i)

Case – II $\quad p + x + y = 0$

$\Rightarrow \quad \frac{dy}{dx} + x + y = 0$

$\Rightarrow \quad \frac{dy}{dx} + y - x$ which is a linear equation.

I.F. = $e^{\int dx} = e^x$

Using the standard result, the solution of the differential equation is :

$$y e^x = -\int xe^x\,dx + A$$

$\Rightarrow \quad y.e^x = e^x (1 - x) + A$

$\Rightarrow \quad y = 1 - x + Ae^{-x}$ ....(ii)

From (i) and (ii), we get the combined solution of the given equation as $(y - Ae^x)(y + x - 1 - Ae^{-x}) = 0$

Example : 16

Solve the differential equation : $y = (1 + p)x + ap^2$, where $p = \frac{dy}{dx}$

Solution

The given differential equation is : $y = (1 + p)x + ap^2$ [solvable for y, refer section 3.3] ....(i)

Differentiating the given equation w.r.t. x, we get

$$\frac{dy}{dx} = p = 1 + p + x\frac{dp}{dx} + 2ap\frac{dp}{dx}$$

$$\Rightarrow \quad 0 = 1 + \frac{dp}{dx}(x + 2ap)$$

$$\Rightarrow \quad \frac{dx}{dp} + x + 2ap = 0, \text{ which is a linear equation.}$$

$$\text{I.F.} = e^{\int dp} = e^p$$

Using the standard result, the solution of the differential equation is :

$$x\,e^p = -2a\int pe^p\,dp + C = -2a(p-1)\,e^p + C$$

$\Rightarrow \quad x = 2a(1 - p) + Ce^{-p}$ ....(ii)

The p-eliminant of (i) and (ii) is the required solution.

Example : 17

Solve the differential equation : $p^2y + 2px = y$

Solution

The given differential is : $x = \frac{y}{2p} - \frac{yp}{2}$ [solvable for x, refer section 3.4] ....(i)

Differentiating with respect to y, we get

$$\frac{dx}{dy} = \frac{1}{p} = \frac{1}{2p} - \frac{y}{2p^2}\frac{dp}{dy} - \frac{p}{2} - \frac{y}{2}\frac{dp}{dy}$$

$$\Rightarrow \quad \frac{1}{2p} + \frac{p}{2} = -\frac{y}{2}\frac{dp}{dy}\left(\frac{1}{p^2} + 1\right)$$

$$\Rightarrow \quad \frac{1+p^2}{2p} = -\frac{y}{2}\frac{dp}{dy}\frac{1+p^2}{p^2}$$

$$\Rightarrow \quad 1 = -\frac{y}{p}\frac{dp}{dy} \quad \text{as } 1 + p^2 \neq 0$$

$\Rightarrow \quad d\,py + ydp = 0$

$\Rightarrow \quad d(py) = 0$

Integrating, we get $py = k \quad \Rightarrow \quad p = \frac{C}{y}$

Putting the value of p in (i), we get

$$y\,.\,\frac{C^2}{y^2} + 2x\,.\,\frac{C}{y} = y$$

$C^2 + 2Cx = y^2$

which si the required solution.

Example : 18

Solve the differential equation : $x = yp + ap^2$.

Solution

The given differential is : $x = yp + ap^2$

Differentiating with respect to y, we get

$$\frac{dx}{dy} = \frac{1}{p} = p + y\frac{dp}{dy} + 2ap\frac{dp}{dy}$$

i.e. $$\frac{1}{p} - p = \frac{dp}{dy}(y + 2ap)$$

i.e. $$\frac{dy}{dp} = \frac{py}{1-p^2} + \frac{2ap^2}{1-p^2}$$

i.e. $$\frac{dy}{dp} - \frac{p}{1-p^2}y = \frac{2ap^2}{1-p^2}$$ which is linear equation

I.F. = $e^{-\int\frac{p}{1-p^2}} = e^{\frac{1}{2}\log(1-p^2)}$

Using the standard result, the solution of the differential equation is :

$$y\sqrt{1-p^2} = 2a\int\frac{p^2}{1-p^2}\cdot\sqrt{1-p^2}\,dp$$

$$= 2a\int\frac{p^2dp}{1-p^2} = -2a\int\frac{(1-p^2)-1}{\sqrt{1-p^2}}\,dp$$

$$= -2a\int\sqrt{1-p^2}\,dp + 2a\int\frac{dp}{\sqrt{1-p^2}}$$

$$= -2a\left[\frac{1}{2}p\sqrt{1-p^2} + \frac{1}{2}\sin^{-1}p\right] + 2a\sin^{-1}p + k$$

$$= y\sqrt{1-p^2} = -ap\sqrt{1-p^2} + a\sin^{-1}p + k. \quad ....(ii)$$

The p-eliminant of (i) and (ii) is the required solution.

Example : 19

Solve the differential equation : $p^3x - p^2y - 1 = 0$

Solution

The given differential equation is : $y = px - 1/p^2$

Differentiating with respect to x, we get

$$\frac{dy}{dx} = p = p + x\frac{dp}{dx} + \frac{2}{p^3}\frac{dp}{dx}$$

$$\Rightarrow \quad \frac{dp}{dx}\left(x + \frac{2}{p^3}\right) = 0$$

$\Rightarrow \quad \frac{dp}{dx} = 0 \quad$ ....(ii)

or $\quad p^3 = \frac{-2}{x} \quad$ ....(iii)

Consider (2)

Integrate both sides to get : p = c where c is an arbitrary constant

put p = c in (i) to get the general solution of the differential equation i.e.

$y = cx - 1/c^2$ is the general solution

Consider (3)

Eliminate p between (iii) and (i) to get the singular solution i.e.

$$y = \frac{\left(\frac{-2}{x}\right)x - 1}{\left(\frac{-2}{x}\right)^{2/3}} = \frac{-3}{\left(\frac{-2}{x}\right)^{2/3}}$$

Take cube of both sides to get : $y^3 = \frac{-27}{4/x^2}$

$\Rightarrow \quad 4y^3 = -2yx^2$ is the singular solution.

Example : 20

Form the differential equation satisfied by the general circle $x^2 + y^2 + 2gx + 2fy + c = 0$

Solution

In forming differential equations for curve, we have to eliminate the arbitrary constants (g, f, v) for n arbitrary constant, we get will finally get an nth order differential equation. Here we will get a third order differential equation in this example.

Differentiating once, $\quad 2x + 2yy' + 2g + 2fy' = 0 \quad$ ....(i)

Differentiating again $\quad 1 + y'^2 + yy'' + fy'' = 0 \quad$ ....(ii)

Differentiating again $\quad 2y'y'' + yy''' + y'y'' + fy''' = 0$

We can now eliminate from (i) and (ii)

$\Rightarrow \quad y'''(1 + yy'' + y'^2) - y''(yy''' + 3y'y'') = 0$

$\Rightarrow \quad y'''(1 + y'^2) - 3y'y''^2 = 0$ is the required differential equation

Example : 21

Find the differential equation satisfied by : $ax^2 + by^2 = 1$

Solution

The given solution is : $ax^2 + by^2 = 1$

Differentiate the above solution to get :

$2ax + 2byy' = 0 \quad$ ....(i)

Differentiating again, we get

$2a + 2b(y'^2 + y'') = 0 \quad$ ....(ii)

Eliminating a and b from (i) and (ii), we will get the required differential equation

from (i), we have $\quad \frac{a}{b} = -\frac{yy'}{x} \quad$ and

from (ii), we have $\quad \frac{a}{b} = -(y'^2 + yy'')$

$\Rightarrow \quad -\frac{yy'}{x} = -(y'^2 + yy'')$

$\Rightarrow \quad yy' = xy'^2 + xyy''$

$\Rightarrow \quad xyy'' + xy'2 - yy' = 0$ is the required differential equation.

Example : 22

The slope of curve passing through (4, 3) at any point is reciprocal of twice the ordinate at that point. Show that the curve is a parabola.

Solution

The slope of the curve is the reciprocal of twice the ordinate at each point of the curve. Using this property, we can define the differential equation of the curve i.e.

$$\text{slope} = \frac{dy}{dx} = \frac{1}{2y}$$

Integrate both sides to get :

$$\int 2y\,dy = \int dx$$

$\Rightarrow$ $y^2 = x + C$

As the required curve passes through (4, 3), it lies on it.

$\Rightarrow$ $9 = 4 + C$ $\Rightarrow$ $C = 5$

So the required curve is : $y^2 = x + 5$ which is a parabola

Example : 23

Find the equation of the curve passing through (2, 1) which has constant subtangent.

Solution

The length of subtangent is constant. Using this property, we can define the differential equation of the curve i.e.

$$\text{subtangent} = \frac{y}{y'} = k \quad \text{where k is a constant}$$

$$\Rightarrow \quad k\frac{dy}{dx} = y$$

Integrate both sides to get :

$$\int \frac{k\,dy}{y} = \int dx$$

$\Rightarrow$ $k \log y = x + C$ where C is an arbitrary constant.

As the required curve passes through (2, 1), it lies on it.

$\Rightarrow$ $0 = 2 + k$ $\Rightarrow$ $C = -2$

$\Rightarrow$ the equation of the curve is : $k \log y = x - 2$.

Note that above equation can also be put in the form $y = Ae^{Bx}$.

Example : 24

Find the curve through (2, 0) so that the segment of tangent between point of tangency and y-axis has a constant length equal to 2

Solution

The segment of the tangent between the point of tangency and y-axis has a constant length = PT = 2. Using this property, we can define the differential equation of the curve i.e.

$$PT = x \sec\theta = x\sqrt{1+\tan^2\theta} = x\sqrt{1+y'^2}$$

$$\Rightarrow \quad x\sqrt{1+\left(\frac{dy}{dx}\right)^2} = 2$$

$$\Rightarrow \quad 1 = \left(\frac{dy}{dx}\right)^2 = \frac{4}{x^2}$$

$$\Rightarrow \quad \frac{dy}{dx} = \pm\sqrt{\frac{4-x^2}{x^2}}$$

Integrate both sides to get :

$$\Rightarrow \quad y = \pm \int \sqrt{\frac{4 - x^2}{x^2}}\ dx + C_1$$

Put x = 2 sin θ ⇒ dx = 2 cos θ dθ

$$\Rightarrow \quad y = 2 \pm \int \frac{\cos^2\theta}{\sin\theta}\ d\theta + C_1 =$$

$$\pm 2 \int (\operatorname{cosec}\theta - \sin\theta)\ d\theta + C_1 = \pm (2 \log |\operatorname{cosec}\theta - \cot\theta| + 2\cos\theta) + C$$

$$\Rightarrow \quad y = \pm 2 \log \left(\left| \frac{2 - \sqrt{4 - x^2}}{x} \right| + \sqrt{4 - x^2} \right) + C$$

As (2, 0) lies on the curve, it should satisfy its equation, i.e. $C = 0$

$$\Rightarrow \quad \text{the equation of the curve is : } y = \pm 2 \log \left(\left| \frac{2 - \sqrt{4 - x^2}}{x} \right| + \sqrt{4 - x^2} \right)$$

Example : 25

Find the equation of the curve passing through the origin if the mid-point of the segment of the normal drawn at any point of the curve and the X-axis lies on the parabola $2y^2 = x$.

Solution

$OB = OM + MB = x + y \tan\theta = x + yy'$

$\Rightarrow \quad B \equiv (x + yy', 0)$

$$\Rightarrow \quad N \text{ (mid point of PB)} \equiv \left(x + \frac{yy'}{2}, \frac{y}{2} \right)$$

N lies on $2y^2 = x$

$$\Rightarrow \quad 2\left(\frac{y}{2}\right)^2 = x + \frac{yy'}{2}$$

$\Rightarrow \quad yy' - y^2 = -2x$ (Divide both sides by y and check that it is a Bernoulli’s differential equation)

$$\text{Put} \quad y^2 = t \quad \Rightarrow \quad 2yy' = \frac{dt}{dx}$$

$$\Rightarrow \quad \frac{1}{2}\frac{dt}{dx} - t = -2x$$

$$\Rightarrow \quad \frac{dt}{dx} - 2t = -4x$$ which is a linear differential equation.

I.F. = Integrating factor = $e^{\int -2\,dx} = e^{-2x}$

Using the standard result, the solution of the differential equation is ;

$$te^{-2x} = \int -4x^{-2x} dx$$

$$\Rightarrow \quad te^{-2x} = -\left(\frac{xe^{-2x}}{-2} + \int \frac{e^{-2x}}{2} dx \right)$$

$$\Rightarrow \quad te^{-2x} = -4\left\{ -\frac{xe^{-2x}}{2} - \frac{e^{-2x}}{4} \right\} + C$$

$$\Rightarrow \quad t = 2x + 1 + Ce^{2x}$$

⇒ $y^2 = 2x + 1 + Ce^{2x}$

As it passes through (0, 0), C = – 1

⇒ $y^2 = 2x + 1 - e^{2x}$ is the required curve.

Example : 26

Find equation of curves which intersect the hyperbola xy = 4 at an angle $\pi/2$.

Solution

Let $m_1 = \frac{dy}{dx}$ for the required family of curves at (x, y)

Let m_2 = value of $\frac{dy}{dx}$ for xy = 4 curve.

⇒ $m_2 = \frac{dy}{dx} = -\frac{4}{x^2}$

As the requied family is perpendicular to the given curve, we can have :

$m_1 \times m_2 = -1$

⇒ $\frac{dy}{dx} \times \left(-\frac{4}{x^2}\right) = -1$

⇒ for required family of curves : $\frac{dy}{dx} = \frac{x^2}{4}$

⇒ $dy = \frac{x^2 dx}{4}$

⇒ $y = \frac{x^3}{12} + C$ is the requied family which intersects xy = 4 curve at an angle $\pi/2$

Example : 27

Solve the differential equation : $(1 + e^{x/y})\, dx = e^{x/y}\left(1 - \frac{x}{y}\right) dy = 0$

Solution

The given differential equation is : $(1 + e^{x/y})\, dx = e^{x/y}\left(1 - \frac{x}{y}\right) dy = 0$ which is a homogenous differential equation.

Put x = my ⇒ $\frac{dx}{dy} = m + y\frac{dm}{dy}$

The given equation reduces to $(1 + e^m)\left(m + y\frac{dm}{dy}\right) + e^m(1 - m) = 0$

$(m + me^m + e^m - me^m) = -(1 + e^m)\, y\frac{dm}{dy}$ ⇒ $\frac{dy}{y} = -\frac{1 + e^m}{m + e^m}\, dm$

Integrating both sides, we get :

$\log y + \log(m + e^m) = C_1$

⇒ $\log y\left(\frac{x}{y} + e^{x/y}\right) = C_1$ ⇒ $x + ye^{x/y} = C$ is the requied general solution.

Example : 28

Solve the equation : $\left(1+x\sqrt{x^2+y^2}\right)$ dx + $\left(-1+\sqrt{x^2+y^2}\right)$ y dy = 0

Solution

The given differential equation can be written as :

$$dx - ydy + x\sqrt{x^2+y^2}\ dx + \sqrt{x^2+y^2}\ ydy = 0$$

$$\Rightarrow \quad dx - ydy + \sqrt{x^2+y^2}\ (xdx + ydy) = 0$$

$$\Rightarrow \quad dx - ydy + \frac{1}{2}\sqrt{x^2+y^2}\ d(x^2 + y^2) = 0$$

Integrating both the sides, we get :

$$x - \frac{y^2}{2} + \frac{1}{2}\int \sqrt{t}\ dt + C = 0 \qquad \text{where } t = x^2 + y^2$$

$$\Rightarrow \quad x - \frac{y^2}{2} + \frac{1}{3}(x^2 + y^2)^{3/2} = C$$

Example : 29

Determine the equation of the curve passing through the origin in the form y = f(x), which satisfies the differential equation dy/dx = sin (10 + 6y)

Solution

Let $10x + 6y = m \quad \Rightarrow \quad \frac{dy}{dx} = \frac{1}{6}\left(\frac{dm}{dx} - 10\right)$

So, we get, $\frac{dm}{dx} = 2\ (3 \sin m = 5)$

$$\Rightarrow \quad \int \frac{dm}{2(3\sin m + 5)} = \int dx$$

Put tan m/2 = t and solve integral on LHS to get :

$$\frac{1}{4}\tan^{-1}\left(\frac{5t+3}{4}\right) = x + C$$

As curve passes through (0, 0) $C = \frac{1}{4}\tan^{-1}\frac{3}{4}$

$$\Rightarrow \quad \tan(4x + \tan^{-1}3/4) = \frac{5\tan(5x+3y)+3}{4}$$

Simplify to get :

$$y = \frac{1}{3}\tan^{-1}\left(\frac{5\tan 4x}{4-3\tan 4x}\right) - \frac{5x}{3} \qquad \left[\text{use } \tan(A+B) = \frac{\tan A + \tan B}{1-\tan A \tan B}\right]$$

Example : 30

Solve the differential equation : $(xy^4 + y)\,dx - x\,dy = 0$

Solution

The given differential equation is : $(xy^4 + y)\,dx - x\,dy = 0$

$$\Rightarrow \quad x\frac{dy}{dx} = xy^4 + y$$

$$\Rightarrow \quad \frac{dy}{dx} - \frac{y}{x} = y^4 \qquad \text{(Bernoulli's differential equation)}$$

Divide both sides by y^4 to get :

$$\frac{1}{y^4}\ \frac{dy}{dx} - \frac{1}{y^3x} = 1 \qquad ....(i)$$

Let $\frac{1}{y^3} = t \quad \Rightarrow \quad \frac{-3}{y^4}\frac{dy}{dx} = \frac{dt}{dx}$

After substitution, (i) reduces to :

$\frac{dt}{dx} + \frac{3t}{x} = -3$ (linear differential equation)

I.F. $e^{\int P\,dx} = e^{\int \frac{3}{x} dx} = e^{3\ell nx} = x^3$

Using the standard result, the solution of differential equation is :

$$tx^3 = \int -3x^3 dx + C_1$$

$$\Rightarrow \qquad tx^3 = \frac{-3x^4}{4} + C$$

$$\Rightarrow \qquad \frac{x^3}{y^3} = -\frac{3}{4}x^4 + C$$

$\Rightarrow \qquad \frac{x^3}{3y^3} + \frac{1}{4}x^4 = C$ is the required general solution.

Alternate Method

Consider the given differential equation, $(xy^4 + y)\,dx - x\,dy = 0$

$\Rightarrow \qquad dy^4dx + ydx - xdy = 0$

Divide both sides by y^4 to get

$$xdx + \frac{ydx - xdy}{y^4} = 0$$

Multiply both sides by x^2 to get :

$$x^2\,dx + \left(\frac{x^2}{y^2}\right)\frac{ydx - xdy}{y^2} = 0$$

$$\Rightarrow \qquad x^3\,dx + \frac{x^2}{y^2}\ d\left[\frac{x}{y}\right] = 0$$

Integrate both sides

$$\int x^3\,dx + \int \frac{x^2}{y^2} d\left(\frac{x}{y}\right) = 0$$

$\Rightarrow \qquad \frac{x^4}{4} + \frac{x^3}{3y^3} = C$ is the requied general solution

Example : 31

Solve the following differential equation : $\frac{xdx + ydy}{xdy - ydx} = \frac{\sqrt{1-(x^2+y^2)}}{\sqrt{x^2+y^2}}$

Solution

The given differential equation can be written as

$$\frac{xdx + ydy}{\sqrt{1-(x^2+y^2)}} = \frac{xdy - ydx}{\sqrt{x^2+y^2}}$$

Divide both sides by $\sqrt{x^2+y^2}$ to get

$$\frac{xdx+ydy}{\sqrt{x^2+y^2}\sqrt{1-(x^2+y^2)}} = \frac{xdy-ydx}{x^2+y^2}$$

Using the fact that d [x^2 + y^2] = 2 (xdx + ydy) and d $\left[\tan^{-1}\frac{y}{x}\right] = \frac{xdy-ydx}{x^2+y^2}$, we get

$$\frac{\frac{1}{2}d\,(x^2+y^2)}{\sqrt{x^2+y^2}\sqrt{1-(x^2+y^2)}} = d\left[\tan^{-1}\frac{y}{x}\right]$$

Put $x^2 + y^2 = t^2$ in the LHS to get :

$$\frac{tdt}{t\sqrt{1-t^2}} = d\left(\tan^{-1}\frac{y}{x}\right)$$

Integrate both sides

$$\int\frac{tdt}{t\sqrt{1-t^2}} = \tan^{-1}\frac{y}{x} + C_1$$

$\Rightarrow$ $\sin^{-1} t = \tan^{-1} (y/x) + C$

so the general solution is : $\sin^{-1}\sqrt{x^2+y^2} = \tan^{-1}\frac{y}{x} + C$

Example : 32

Solve the differential equation : $\frac{dy}{dx} + x \sin 2y = x^3 \cos^2 y$.

Solution

The given differential equation is : $\frac{dy}{dx} + x \sin 2y = x^3 \cos^2 y$

Dividing both sides by $\cos^2 y$, we get

$$\sec^2 y\frac{dy}{dx} + 2x \tan y = x^3$$

Let $\tan y = t$ $\Rightarrow$ $\sec^2 y\frac{dy}{dx} = \frac{dt}{dx}$

On substitution, differential equation reduces to :

$\frac{dt}{dx} + 2xt = x^3$ (linear differential equation)

I.F. = $e^{\int 2x\,dx} = e^{x^2}$

Using the standard result, the general solution is :

$$te^{x^2} = \int x^3e^{x^2}dx + C_1$$

Integrate RHS yourself to get the general solution :

$$te^{x^2} = \frac{1}{2}(x^2-1)\, e^{x^2} + C$$

Replace t by tan y, we get :

$\tan y = \frac{1}{2}(x^2-1)\, Ce^{-x^2}$ which is the requied solution

Example : 33

A normal is drawn at a point P(x, y) of a curve. It meets the x-axis at Q. If PQ is of constant length k, then show that the differential equation describing such curves is $y\dfrac{dy}{dx} = \pm\sqrt{k^2 - y^2}$. Also find the equation of the curve if it passes through (0, k) point

Solution

Let M be the foot of the perpendicular drawn from P to the x-axis
In triangle PMQ,
PQ = k (given), QM = subnormal = y (dy/dx) and PM = y
Apply pythagoras theorem in triangle PMQ to get :

$$PQ^2 = PM^2 + MQ^2$$

$$\Rightarrow \quad k^2 = y^2 + y^3\left(\frac{dy}{dx}\right)^2$$

$$\Rightarrow \quad y\frac{dy}{dx} = \pm\sqrt{k^2 - y^2}$$ which is requied to be shown

Solving the above differential equation, we get :

$$\int \frac{y\,dy}{\sqrt{k^2 - y^2}} = \pm\int dx$$

$$\Rightarrow \quad -\sqrt{k^2 - y^2} = \pm x + C$$

As (0, k) lies on it, 0 = 0 + C $\Rightarrow$ C = 0

$\Rightarrow$ equation of curve is : $-\sqrt{k^2 - y^2} = \pm x$

$\Rightarrow$ $x^2 + y^2 = k^2$ is the required equation of the curve.

Example : 34

A curve y = f(x) passes through the point P(1, 1). The normal to the curve at P is : a(y – 1) + (x – 1) = 0. If the slope of the tangent at any point on the curve is proportional to the ordinate of that point, determine the equation of the curve. Also obtain the area bounded by the y-axis, the curve and the normal to the curve at P.

Solution

It is given that equation of the normal at point P(1, 1) is ≡ ay + x = a + 1

$\Rightarrow$ slope of tangent at P = –1/(slope of normal at P)

$$\Rightarrow \quad \left.\frac{dy}{dx}\right]_{\text{at P}} = a \qquad \text{...........(i)}$$

It is also given that slope of the tangent at any point of the curve is proportional to the ordinate i.e.

$$\Rightarrow \quad \tan\theta = \frac{dy}{dx} = dy$$

$$\Rightarrow \quad \frac{dy}{dx} = ay \qquad [\because \text{ from (i0, at P(1, 1), dy/dx = a]}$$

On solving, we get : $\ell nx = ax + C$

As curve passes through (1, 1), 0 = a + C $\Rightarrow$ C = – a

$\Rightarrow$ equation of the curve is : $y = e^{x(x-1)}$

$$\text{requied Area} = \int_0^1 \left[\frac{1-x}{a} + 1 - e^{x(x-1)}\right] dx = \left|\frac{x}{a} - \frac{x^2}{2a} + x - \frac{e^{x(x-1)}}{a}\right|_0^1$$

$$= \left(\frac{1}{a} - \frac{1}{2a} + 1 - \frac{1}{a}\right) + \frac{e^{-a}}{a} = \frac{2e^{-a} - 1 + 2a}{2a}$$

Example : 35

Find the equation to the curve such that the distance between the origin and the tangent at an arbitrary point is equal to the distance between the origin and the normal at the same point.

Solution

Equation of tangent to the curve y = f(x) and any point (x, y) is :

$$Y - y = f'(x)\,(X - x) \qquad \text{.............(i)}$$

The distance of the tangent from origin = $\dfrac{|\,y - f'(x)\,x\,|}{\sqrt{1+(f'(x))^2}}$ ....(i)

Equation of norma to the curve y = f(x) and any point (x, y) is :

$$Y - y = -\frac{1}{f'(x)}\,(X - x)$$

The distance of the normal from origin = $\dfrac{\left|y+\dfrac{1}{f'(x)}x\right|}{\sqrt{1+\left(\dfrac{1}{f'(x)}\right)^2}}$ ....(ii)

From (i) and (ii) and using the fact that the distance of the tangent and normal from origin is equal, we get:

$$y - f'(x)\,x = f'(x)\left|\,y+\frac{1}{f'(x)}x\,\right| = \pm\,[f'(x)\,y + x]$$

$$\Rightarrow \qquad y - x = (x + y)\,\frac{dy}{dx} \qquad \text{or} \qquad x + y = (x - y)\,\frac{dy}{dx}$$

$$\Rightarrow \qquad \frac{dy}{dx} = \frac{y-x}{y+x} \qquad \text{or} \qquad \frac{dy}{dx} = \frac{x+y}{x-y}$$

Consider case – I

$\dfrac{dy}{dx} = \dfrac{y-x}{y+x} = \dfrac{y/x-1}{y/x+1}$ which is a homogeneous equation.

Put $y = mx$ $\Rightarrow$ $dy/dx = m + x\,(dm/dx)$

On substituting in the differential equation, we get :

$$m + x\,\frac{dm}{dx} = \frac{m-1}{m+1}$$

$$\Rightarrow \qquad \frac{dx}{x} = -\left(\frac{1+m}{1+m^2}\right) dm$$

Integrate both sides, to get :

$$\int\frac{dx}{x} = \int\left(-\frac{1}{1+m^2} - \frac{1}{2}\cdot\frac{2m}{1+m^2}\right) dm$$

$$\Rightarrow \qquad \log x = -\tan^{-1} m - 1/2 \log (1 + m^2) + C$$

$$\Rightarrow \qquad \log x\,(1 + m^2)^{1/2} = -\tan^{-1} m + C$$

$$\Rightarrow \qquad x\left(1+\frac{y^2}{x^2}\right)^{1/2} = Ce^{-\tan^{-1} y/x}$$

$\Rightarrow$ $\sqrt{x^2+y^2} = Ce^{-\tan^{-1} y/x}$ is the general solution

Consider case – II

$\dfrac{dy}{dx} = \dfrac{x+y}{x-y} = \dfrac{1+y/x}{1-y/x}$ which is a homogeneous equation.

On solving the above homogenous differential equation, we can get :

$\sqrt{x^2+y^2} = Ce^{\tan^{-1}y/x}$ as the general solution

Example : 36

Show that curve such that the ratio of the distance between the normal at any of its points and the origin to the distance between the same normal and the point (a, b) si equal to the constant k(k > 0) is a circle if $k \neq 1$.

Solution

Equation of the normal at any point (x, y) to curve y = f(x) is

$$Y - y = -\frac{1}{f'(x)}(X - x)$$

its distance from origin = $\dfrac{\left|y+\dfrac{x}{f'(x)}\right|}{\sqrt{1+\left(\dfrac{1}{f'(x)}\right)^2}}$

The distance of the normal from (a, b) = $\dfrac{\left|y-b\dfrac{1}{f'(x)}(x-a)\right|}{\sqrt{1+\left(\dfrac{1}{f'(x)}\right)^2}}$

As the ratio of these distances is k, we get :

$$\left|y+\frac{x}{f'(x)}\right| = k\left|y-b+\frac{1}{f'(x)}(x-a)\right|$$

$$y+\frac{x}{f'(x)} = \pm k\left(y-b+\frac{1}{f'(x)}(x-a)\right)$$

$(1 - k)\,y + bk = (kx - x - ak)\dfrac{dx}{dy}$ and $(1 + k)\,y - bk = (-kx - x + ak)\dfrac{dx}{dy}$

$\Rightarrow$ $(1 - k)\,ydy + bkdy = kxdx - xdx - akdx$ and $(1 + k)\,ydy - bkdy = -kxdx - xdx + akdx$

Integrating both the sides

$(1 - k)\dfrac{y^2}{2} = bky = \left(k\dfrac{x^2}{2}-\dfrac{x^2}{2}-akx\right) + C_1$ and $(1 + k)\dfrac{y^2}{2} - bky = \left(-k\dfrac{x^2}{2}-\dfrac{x^2}{2}+akx\right) + C_2$

$\dfrac{(1-k)}{2}x^2 + (1 - k)\dfrac{y^2}{2} + bky + akx + C_1 = 0$ and $\dfrac{(1-k)}{2}x^2 + (1 + k)\dfrac{y^2}{2} - bky - akx + C_2 = 0$

If $k \neq 1$, then both the above equations represent circle.

Example : 37

Let y = f(x) be a curve passing through (1, 1) such that the triangle formed by the coordinate axes and the tangent at any point of the curve lies in the first quadrant and has area 2. From the differential equation and determine all such possible curves.

Solution

Equation of tangent at (x, y) = $Y' - y = \dfrac{dy}{dx}(X - x)$

$X_{\text{intercept}} = x - \dfrac{y}{dy/dx}$ and $Y_{\text{intercept}} = y - x\dfrac{dy}{dx}$

Area of the triangle = $\left|\frac{1}{2} X_{\text{int ercept}} \times Y_{\text{int ercept}}\right|$ = 2

Both X-intercept and Y-intercept are positive as the triangle lies in the first quadrant. So we can remove mod sign.

$\Rightarrow$ $\left(x - \frac{y}{y'}\right)$ $(y - xy') = 4$

$\Rightarrow$ $(xy' - y)^2 = -4y'$

$\Rightarrow$ $xy' - y = -2\sqrt{-y'}$ $\left(\because\ y_{\text{int}} = y - \frac{xdy}{dx} > 0 \quad \Rightarrow \quad xy' - y < 0\right)$

$\Rightarrow$ $y = xy' + 2\sqrt{-y'}$ (Clairaut's differential equation) ....(i)

Differentiate both sides w.r.t. to x, to get :

$\Rightarrow$ $y' = xy'' + y' + \frac{2}{2\sqrt{-y'}}\,(-y'')$

$\Rightarrow$ $y'' = 0$ or $x = \frac{1}{\sqrt{-y'}}$

consider $y'' = 0$ integrate both sides to get : $y' = c$

Put $y' = c$ in (i) to get the general solution of the equation i.e.

$y = cx + 2\ 2\sqrt{-c}$

As the curve passes through (1, 1), c = – 1 (check yourself)

$\Rightarrow$ the equation of the curve is : $x + y = 2$

Consider : $x = \frac{1}{\sqrt{-y'}}$

$\Rightarrow$ $y' = \frac{-1}{x^2}$ ....(ii)

To find singular solution of the Clairaut's equation, eliminate y' in (i) and (ii)

Replace y' from (ii) into (i) to get :

$$y = \frac{-x}{x^2} + 2\sqrt{\frac{1}{x^2}} = \frac{-1}{x} + \frac{2}{x} = \frac{1}{x}$$

$\Rightarrow$ the requied curves are $y = 1/x$ and $x + y = 2$.

Example : 38

Let u(x) and v(x) satisfy the differential equations $\frac{du}{dx} + P(x)\,u = f(x)$ and $\frac{dv}{dx} + P(x)\,v = g(x)$ where P(x), f(x) and g(x) are continuous function. If $u(x_1) > v(x_1)$ for some x_1 and $f(x) > g(x)$ for all $x > x_1$, prove that any point (x, y) where $x > x_1$

Solution

The given differential equation are :

$\frac{du}{dx} = P(x) = u = f(x)$ ....(i)

$\frac{dv}{dx} = P(x)\,v = g(x)$ ....(ii)

On subtracting the two differential equations, we get

$\frac{d}{dx}(u - v) + P(x)\,(u - v) = f(x) - g(x)$

For $x > x_1$, $f(x) > g(x)$ $\Rightarrow$ $\frac{d}{dx}(u - v) + P(x)(u - v) > 0$

$\Rightarrow$ $\frac{d(u-v)}{u-v} > -P(x)\, dx$

Integrate both sides to get :

$$\ell n\,(u - v) + C > \int -P(x)\, dx$$

$\Rightarrow$ $u - v > e^{\int P(x)\, dx - C}$

As RHS > 0 for all x, u > v for all $x > x_1$

$\Rightarrow$ y = u(x) and y = v(x) have no solution (i.e. no point of intersection as one curve lies above the other)

Example : 39

A and B are two separate reservoirs of water. Capacity of reservoir A is double the capacity of reservoir B. Both the reservoirs are filled completely with water, their inlets are closed and then the water is released simultaneously from both the reservoirs. The rate of flow of water out of each reservoir at any instant of time is proportional to the quantity of water in the reservoir at that time. One hour after the water is released, the quantity of water in reservoir is $1\frac{1}{2}$ times the quantity of water in reservoir B. After how many hours do both the reservoir have the same quantity of water?

Solution

Let V_{Ai} and V_{Bi} be the initial amounts of water in reservoirs A and B respectively

As capacity of reservoir A si double that of B and both are completely filled initially, we can have:

$$V_{Ai} = 2V_{Bi}$$

Let V_A and V_n be the amount of water in reservoirs A and B respectively at any instant fo time t.

As the rate of flow of water out of each reservoir at any instant of time is proportional to the quantity of water in the reservoir at that time, we can have :

$$\frac{dV_A}{dt} = -k_1 V_A \qquad \text{............(i)}$$

and $$\frac{dV_B}{dt} = -k_2 V_B \qquad \text{............(ii)}$$

where k_1 and k_2 are proportionality constants.

Let V_{Af} and V_{Bf} be the amounts of water in reservoirs A and B respectively after 1 hour.

To find V_{Ar} and A_{bf} integrate (i) and (ii)

$$\Rightarrow \int_{V_{Ai}}^{V_{Af}} \frac{dV_A}{V_A} = -\int_0^1 k_1 dt \quad \Rightarrow \quad \ell n\left(\frac{V_{Af}}{V_{Ai}}\right) = -k_1$$

Similarity we can get : $\ell n\left(\frac{V_{Bf}}{V_{Bi}}\right) = -k_2$ $\Rightarrow$ $V_{Ai}\, e^{-k_1} = \frac{3}{2} V_{Bi}\, e^{-k_2}$

$$\Rightarrow \quad k_1 - k_2 = \ell n\left(\frac{4}{3}\right) \qquad \text{............(iii)}$$

After time t $V_A = V_B$

$\Rightarrow$ $V_{Ai}e^{-k_1 t} = V_{Bi}e^{-k_2 t}$

$\Rightarrow$ $2e^{-k_1 t} = e^{-k_2 t}$

$$\Rightarrow \quad (k_1 - k_2)\, t = \ell n\, 2 \qquad \text{............(iv)}$$

Solving (iii) and (iv), we get : $t = \dfrac{\ell n 2}{\ell n\left(\frac{4}{3}\right)}$