Example : 1

Discuss the differentiability of f(x) at x = – 1, if f (x) = $\begin{cases} 1-x^2 & ; \quad x \le -1 \\ 2x+2 & ; \quad x > -1 \end{cases}$

Solution

$f(-1) = 1 - (1)^2 = 0$

Right hand derivative at x = – 1 is

$$Rf'(-1) = \lim_{h\to 0} \frac{f(-1+h)-f(-1)}{h} = \lim_{h\to 0} \frac{2(-1+h)+2-0}{h} = \lim_{h\to 0} \frac{2h}{h} = 2$$

Left hand derivative at x = – 1 is

$$Lf'(-1) = \lim_{h\to 0} \frac{f(-1-h)-f(-1)}{-h} = \lim_{h\to 0} \frac{1-(-1-h)^2-0}{-h} = \lim_{h\to 0} \frac{-h^2-2h}{-h} = \lim_{h\to 0} (h+2) = 2$$

Hence Lf′ (–1) = Rf′ (–1) = 2

⇒ the function is differentiable at x = – 1

Example : 2

Show that the function : $f(x) = |x^2 - 4|$ is not differentiable at x = 2

Solution

$$f(x) = \begin{cases} x^2-4 & ; \quad x \le -2 \\ 4-x^2 & ; \quad -2 < x < 2 \\ x^2-4 & ; \quad x \ge 2 \end{cases} \quad \Rightarrow \quad f(2) = 2^2 - 4 = 0$$

$$Lf'(2) = \lim_{h\to 0} \frac{f(2-h)-f(2)}{-h} = \lim_{h\to 0} \frac{4-(2-h)^2-0}{-h} = \lim_{h\to 0} \frac{4h-h^2}{-h} = \lim_{h\to 0} (h-4) = -4$$

$$Rf'(2) = \lim_{h\to 0} \frac{f(2+h)-f(2)}{h} = \lim_{h\to 0} \frac{[(2+h)^2-4]-0}{h} = \lim_{h\to 0} \frac{h^2+4h}{h} = \lim_{h\to 0} (h+4) = 4$$

⇒ Lf′(2) ≠ Rf′(2)

Hence f(x) is not differentiable at x = 2

Example : 3

Show that f(x) = x |x| is differentiable at x = 0

Solution

$$f(x) = \begin{cases} -x^2 & ; \quad x \le 0 \\ x^2 & ; \quad x > 0 \end{cases}$$

$$Lf'(0) = \lim_{h\to 0} \frac{f(0-h)-f(0)}{-h} = \lim_{h\to 0} \frac{-(-h)^2-0}{-h} \lim_{h\to 0} h = 0$$

$$Rf'(0) = \lim_{h\to 0} \frac{f(0+h)^2-f(0)}{h} \lim_{h\to 0} \frac{h^2-0}{h} = 0$$

⇒ Lf′(0) = Rf′(0)

Hence f(x) is differentiable at x = 0

Example : 4

Prove that following theorem :

"If a function y = f(x) is differentiable at a point, then it must be continuous at that point."

Solution

Let the function be differentiable at x = a.

⇒ $\lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$ and $\lim_{h\to 0} \frac{f(a-h)-f(a)}{-h}$ are finite numbers which are equal

L.H.L. = $\lim_{h\to0}$ f (a – h)

= $\lim_{h\to0}$ [f(a – h) – f(a)] + f(a)

= $\lim_{h\to0}$ (–h) $\left[\lim_{h\to0}\frac{f(a-h)-(a)}{-h}\right]$ + f(a)

= 0 × [Lf′(a)] + f(a) = f(a)

R.H.L. = $\lim_{h\to0}$ f(a + h)

= $\lim_{h\to0}$ [f(a + h) – f(a)] + f(a)

= $\lim_{h\to0}$ h $\left[\lim_{h\to0}\frac{f(a+h)-(a)}{h}\right]$ + f(a) = 0 × [Rf′(a)] + f (a) = f(a)

Hence the function is continuous at x = a

Note : that the converse of this theorem is not always true. If a function is continuous at a point, if may or may not be differentiable at that point.

Example : 5

Discuss the continuity and differentiability of f(x) at x = 0 if f(x) = $\begin{cases} x^2\sin\frac{1}{x} & ; \quad x\neq 0 \\ 0 & ; \quad x = 0\end{cases}$

Let us check the differentiability first. Lf′(0) = $\lim_{h\to0}\ \frac{f(0-h)-f(0)}{-h}$ = $\lim_{h\to0}\ \frac{(-h)^2\sin\left(\frac{1}{-h}\right)-0}{-h}$

= $\lim_{h\to0}$ h sin $\frac{1}{h}$ = $\lim_{h\to0}$ h × $\lim_{h\to0}$ sin $\frac{1}{h}$

= 0 × (number between – 1 and + 1) = 0

Rf′(0) = $\lim_{h\to0}\ \frac{f(0+h)-f(0)}{h}$ = $\lim_{h\to0}\ \frac{h^2\sin\frac{1}{h}-0}{h}$ = $\lim_{h\to0}$ h sin $\frac{1}{h}$ = $\lim_{h\to0}$ h × $\lim_{h\to0}$ sin $\frac{1}{h}$

= 0 × (number between – 1 and + 1) = 0

Hence Lf′(0) = Rf′(0) = 0

⇒ function is differentiable at x = 0

⇒ if must be continuous also at the same point.

Example : 6

Show that the function f(x) is continuous at x = 0 but its derivative does not exists at x = 0 if

f(x) = $\begin{cases} x\sin\left(\log x^2\right) & ; \quad x\neq 0 \\ 0 & ; \quad x = 0\end{cases}$

Solution

Test for continuity :

LHL = $\lim_{h\to0}$ f(0 – h) = $\lim_{h\to0}$ (–h) sin log $(-h)^2$ = – $\lim_{h\to0}$ h sin log h^2

as h → 0, log h^2 → – ∞

Hence sin log h^2 oscillates between – 1 and + 1

⇒ LHL = – $\lim_{h\to0}$ (h) × $\lim_{h\to0}$ (sin log h^2) = – 0 × (number between – 1 and + 1) = 0

R.H.L. = $\lim_{h\to0}$ f(0 + h) = $\lim_{h\to0}$ = $\lim_{h\to0}$ h sin logh^2

$= \lim_{h\to 0} h \,.\, \lim_{h\to 0} \sin \log h^2 = 0 \times$ (oscillating between – 1 and + 1) = 0

f(0) = 0 (Given)

$\Rightarrow$ LHL = RHL = f(0)

Hence f(x) is continuous at x = 0

Test for differentiability :

$$Lf'(0) = \lim_{h\to 0} \frac{f(0-h)-f(0)}{-h} = \lim_{h\to 0} \frac{-h\sin\log(-h)^2 - 0}{-h} = \lim_{h\to 0} \sin(\log h^2)$$

As the expression oscillates between – 1 and + 1, the limit does not exists.

$\Rightarrow$ Left hand derivative is not defined.

Hence the function is not differentiable at x = 0

Note : As LHD is undefined there is no need to check RHD for differentiability as for differentiability both LHD and RHD should be defined and equal

Example : 7

Discuss the continuity of f, f′ and f′′ on [0, 2] if $f(x) = \begin{cases} \dfrac{x^2}{2} & ; \ 0 \le x < 1 \\ 2x^2 - 3x + \dfrac{3}{2} & ; \ 1 \le x \le 2 \end{cases}$

Solution

Continuity of f(x)

For $x \ne 1$, f(x) is a polynomial and hence is continuous

At x = 1, LHL = $\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} \frac{x^2}{2} = \frac{1}{2}$

RHL = $\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} \left(2x^2 - 3x + \frac{3}{2}\right) = 2 - 3 + \frac{3}{2} = \frac{1}{2}$

$f(1) = 2(1)^2 - 3(1) + \frac{3}{2} = \frac{1}{2}$

$\Rightarrow$ LHL = RHL = f(1)

Therefore, f(x) is continuous at x = 1

Continuity of f′(x)

Let g(x) = f′(x)

$\Rightarrow \quad g(x) = \begin{cases} x & ; \ 0 \le x < 1 \\ 4x - 3 & ; \ 1 \le x \le 2 \end{cases}$

For $x \ne 1$, g(x) is linear polynomial and hence continuous.

At x = 1, LHL = $\lim_{x\to 1^-} g(x) = \lim_{x\to 1^-} x = 1$

RHL = $\lim_{x\to 1^-} g(x) = \lim_{x\to 1^-} (4x - 3) = 1$

g(1) = 4 – 3 = 1

$\Rightarrow$ LHL = RHL = g(1)

$\therefore$ g(x) = f′(x) is continuous at x = 1

Continuity of f′′(x)

Let h(x) = f′′(x) = $\begin{cases} 1 & ; \ 0 \le x < 1 \\ 4 & ; \ 1 \le x \le 2 \end{cases}$

For $x \ne 1$, h(x) is continuous because it is a constant function.

At x = 1, LHL = $\lim_{x\to 1^-} h(x) = 1$

RHL = $\lim_{x\to 1^+} h(x) = 4$

Thus LHL ≠ RHL

∴ h(x) is discontinuous at x = 1

Hence f(x) and f′(x) are continuous on [0, 2] but f′′(x) is discontinuous at x = 1.

Note : Continuity of f′(x) is same as differentiability of f(x)

Example : 8

Show that $\lim_{x\to a} \frac{f(x)g(a) - g(x)f(a)}{x - a}$ = f′(a) g(a) – g′(a) f(a) if f(x) and g(x) are differentiable at x = a.

Solution

$$\lim_{x\to a} \frac{f(x)g(a) - g(x)f(a)}{x-a} = \lim_{x\to a} \frac{f(x)g(x) - f(a)g(a) + f(a)g(a) - g(x)f(a)}{x-a}$$

$$= \lim_{x\to a} \left[\frac{f(x) - f(a)}{x-a}\right] g(a) - \lim_{x\to a} \left[\frac{g(x) - g(a)}{x-a}\right] f(a)$$

= f′(a) g(a) – g′ (a) f(a)

Example : 9

Let f(x) be defined in the interval [–2, 2] such that f(x) = $\begin{cases} -1 & ; \ -2 \le x \le 0 \\ x-1 & ; \ 0 < x \le 2 \end{cases}$ and g(x) = f(|x|) + |f(x)|. Test the differentiability of g(x) in (–2, 2).

Solution

Consider f(|x|)

The given interval is – 2 ≤ x ≤ 2

Replace x by |x| to get :

– 2 ≤ |x| ≤ 2 ⇒ 0 ≤ |x| ≤ 2

Hence f(|x|) can be obtained by substituting |x| in place of x in x – 1 [see definition of f(x)].

⇒ f(|x|) = |x| – 1 ; – 2 ≤ x ≤ 2 ....(i)

Consider |f(x)|

Now |f(x)| = $\begin{cases} |-1| & ; \ -2 \le x \le 0 \\ |x-1| & ; \ 0 < x \le 2 \end{cases}$ ⇒ |f(x)| = $\begin{cases} 1 & ; \ -2 \le x \le 0 \\ |x-1| & ; \ 0 < x \le 2 \end{cases}$

adding (i) and (ii)

$$f(|x|) + |f(x)| = \begin{cases} |x| - 1 + 1 & ; \ -2 \le x \le 0 \\ |x| - 1 + |x-1| & ; \ 0 < x \le 2 \end{cases}$$

$$\Rightarrow \quad g(x) = \begin{cases} |x| & ; \ -2 \le x \le 0 \\ |x| - 1 + |x-1| & ; \ 0 < x \le 2 \end{cases}$$

on further simplification,

$$g(x) = \begin{cases} -x & ; \ -2 \le x \le 0 \\ x - 1 + 1 - x & ; \ 0 < x < 1 \\ x - 1 + x - 1 & ; \ 1 \le x \le 2 \end{cases} \qquad g(x) = \begin{cases} -x & ; \ -2 \le x \le 0 \\ 0 & ; \ 0 < x < 1 \\ 2x - 2 & ; \ 1 \le x \le 2 \end{cases}$$

For x ≠ 0 and x ≠ 1, g(x) is a differentiable function because it is a linear polynomial

At x = 0

$$Lg'(0) = \lim_{h\to 0} \frac{g(0-h) - g(0)}{-h} = \lim_{h\to 0} \frac{-(-h) - 0}{-h} = -1$$

$$Rg'(0) = \lim_{h\to 0} \frac{g(0+h) - g(0)}{h} = \lim_{h\to 0} \frac{0-0}{h} = 0$$

⇒ Lg′ (0) ≠ Rg′ (0). Therefore g(x) is not differentiable at x = 0

At x = 1

$$Lg'(1) = \lim_{h\to 0} \frac{g(1-h) - g(1)}{-h} = \lim_{h\to 0} \frac{0-0}{-h} = 0$$

$$Rg'(1) = \lim_{h\to 0} \frac{g(1+h)-g(1)}{h} = \lim_{h\to 0} \frac{2(1+h)-2-0}{h} = 2$$

$\Rightarrow$ $Lg'(1) \neq Rg'(1)$. Therefore g(x) in not differential at x = 1

Hence g(x) is not differentiable at x = 0, 1 in (–2, 2)

Example : 10

Find the derivative of y = log x wrt x from first principles.

Solution

Let f(x) = log x

Using definition of derivative

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

$$\Rightarrow \quad f'(x) = \lim_{h\to 0} \frac{\log(x+h)-\log x}{h} = \lim_{h\to 0} \frac{\log\left(1+\frac{h}{x}\right)}{h} = \lim_{h\to 0} \frac{\log\left(1+\frac{h}{x}\right)}{h/x}\,\frac{1}{x} = \frac{1}{x}$$

$$\left[\text{using } \lim_{t\to 0}\frac{\log(1+t)}{1} = 1\right]$$

Example : 11

Evaluate the derivative f (x) = x^n wrt x from definition of derivative. Hence find the derivative of $\sqrt{x}$, 1/x, 1/$\sqrt{x}$, 1/x^p wrt x.

Solution

Using definition of derivative

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

$$f'(x) = \lim_{h\to 0} \frac{(x+h)^n - x^n}{h} = \lim_{h\to 0} \frac{(x+h)^n - x^n}{(x+h)-x} = \lim_{t\to x} \frac{t^n - x^n}{t-x} \qquad \text{(putting } t = x + h\text{)}$$

$$= nx^{n-1} \left[\text{using } \lim_{x\to a}\frac{x^n - a^n}{x-a} = na^{n-1}\right]$$

Taking n = $\frac{1}{2}$, $\frac{d}{dx}\sqrt{x} = \frac{1}{2\sqrt{x}}$

taking n = – 1, $\frac{d}{dx}\left(\frac{1}{x}\right) = \frac{-1}{x^2}$

taking n = $\frac{-1}{2}$, $\frac{d}{dx}\left(\frac{1}{\sqrt{x}}\right) = \frac{-1}{2x\sqrt{x}}$

taking n = –p, $\frac{d}{dx}\left(\frac{1}{x^p}\right) = \frac{-p}{x^{p+1}}$

Example : 12

Find the derivative of sin x wrt x from first principles.

Solution

Let f(x) = sin x

Using the definition of derivative,

$$f'(x) = \lim_{h\to 0} \frac{\sin(x+h)-\sin x}{h} = \lim_{h\to 0} \frac{2\cos\left(x+\frac{h}{2}\right)\sin\frac{h}{2}}{2\ \frac{h}{2}} = \cos x \ . \lim_{h\to 0} \frac{\sin\frac{h}{2}}{\frac{h}{2}} = \cos x$$

$$\left(\text{using} \lim_{\theta\to 0} \frac{\sin\theta}{\theta} = 1\right)$$

Hence f′(x) = cos x

Example : 13

Differentiate a^x wrt x from first principles

Solution

Let f(x) = a^x

Using the definition of derivatives $f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$

$$\Rightarrow \quad f'(x) = \lim_{h\to 0} \frac{a^{x+h}-a^x}{h} = a^x \ . \lim_{h\to 0} \frac{a^h-1}{h} = a^x \log a \qquad \left(\text{using} \lim_{t\to 0} \frac{a^t-1}{t} = \log a\right)$$

Hence f′(x) = a^x log a

Example : 14

Differentiate sin (log x) wrt x from first principles

Solution

Let f(x) = sin (log x)

Using the definition of derivatives

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

$$f'(x) = \lim_{h\to 0} \frac{\sin\log(x+h)-\sin\log x}{h} = \lim_{h\to 0} \frac{2\cos\left(\frac{\log(x+h)+\log x}{2}\right)\sin\left(\frac{\log(x+h)-\log x}{2}\right)}{h}$$

$$= \lim_{h\to 0} 2\cos\left(\frac{\log(x+h)+\log x}{2}\right) \times \lim_{h\to 0}\left(\frac{\sin\left(\frac{\log(x+h)-\log x}{2}\right)}{h}\right)$$

$$= 2\cos\log x \lim_{h\to 0} \frac{\sin\left(\frac{\log(x+h)-\log x}{2}\right)}{\frac{\log(x+h)-\log x}{2}} \times \lim_{h\to 0} \frac{\log(x+h)-\log x}{2h}$$

$$= 2\cos\log x \ . \ 1 \ . \lim_{h\to 0} \frac{\log(1+h/x)}{2h} \qquad \left[\because \lim_{\theta\to 0} \frac{\sin\theta}{\theta} = 1\right]$$

$$= \cos\log x \ . \lim_{h\to 0} \frac{\log(1+h/x)}{h/x} \ . \ \frac{1}{x} = \frac{\cos\log x}{x} \qquad \left[\because \lim_{t\to 0} \frac{\log(1+t)}{t} = 1\right]$$

Example : 15

Differentiate x^2 tan x wrt x from first principles

Solution

Let f(x) = x^2 tan x

Using the definition of derivative,

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

$$f'(x) = \lim_{h\to 0} \frac{(x+h)^2\tan(x+h)-x^2\tan x}{h} = \lim_{h\to 0} \frac{x^2\tan(x+h)-x^2\tan x+(h^2+2hx)\tan(x+h)}{h}$$

$$= x^2 \lim_{h\to 0} \frac{\tan(x+h)-\tan x}{h} + \lim_{h\to 0} \frac{h(h+2)\tan(x+h)}{h}$$

$$= x^2 \lim_{h\to 0} \frac{\sin(x+h-x)}{h\cos x\cos(x+h)} + \lim_{h\to 0} (h+2x)\tan(x+h)$$

$$= \frac{x^2}{\cos^2 x} + 2x\tan x \qquad \left[\text{using } \lim_{\theta\to 0}\frac{\sin\theta}{\theta}=1\right]$$

Example : 16

Differentiate $\sin^{-1}x$ from first principles

Solution

Let y = $\sin^{-1}x$ $\Rightarrow$ x = sin y

From first principles

$$\frac{dx}{dy} = \lim_{h\to 0} \frac{f(y+h)-f(y)}{h} \Rightarrow \frac{dx}{dy} = \lim_{h\to 0} \frac{\sin(y+h)-\sin y}{h}$$

$$= \lim_{h\to 0} \frac{2\cos\left(\frac{2y+h}{2}\right)\sin\frac{h}{2}}{h} = \frac{dx}{dy} = \cos y \lim_{h\to 0} \frac{\sin\frac{h}{2}}{h/2} = \cos y$$

As (dy/dx) × (dx/dy) = 1, we get

$$\frac{dy}{dx} = \frac{1}{\cos y} = \frac{1}{\pm\sqrt{1-\sin^2 y}} = \frac{1}{\pm\sqrt{1-x^2}} \qquad (\because x = \sin y)$$

But the principal value of y $\sin^{-1}x$ lies between $-\pi/2$ and $\pi/2$ and for these values of y, cos y is positive.
($\because$ cosine of an angle in the first or fourth quadrant is positive)

Therefore rejecting the negative sign, we have $\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}$

Example : 17

Differentiate $\sqrt{\tan\sqrt{x}}$ from first principles.

Solution

Let f(x) = $\sqrt{\tan\sqrt{x}}$

From first principles,

$$f'(x) = \lim_{h\to 0} \frac{\sqrt{\tan\sqrt{x+h}}-\sqrt{\tan\sqrt{x}}}{h}$$

Rationalise to get,

$$f'(x) = \lim_{h\to 0} \frac{\tan\sqrt{x+h}-\tan\sqrt{x}}{h\left(\sqrt{\tan\sqrt{x+h}}+\sqrt{\tan\sqrt{x}}\right)} \Rightarrow f'(x) = \frac{1}{2\sqrt{\tan\sqrt{x}}} \lim_{h\to 0} \frac{\sin\left(\sqrt{x+h}-\sqrt{x}\right)}{h\cos\sqrt{x+h}\cos\sqrt{x}}$$

$$\Rightarrow \quad f'(x) = \frac{1}{2\sqrt{\tan\sqrt{x}}\cos^2\sqrt{x}} \times \lim_{h\to 0} \frac{\sin\left(\sqrt{x+h}-\sqrt{x}\right)\left(\sqrt{x+h}-\sqrt{x}\right)}{\left(\sqrt{x+h}-\sqrt{x}\right)h}$$

$$\Rightarrow \quad f'(x) = \frac{1}{2\sqrt{\tan\sqrt{x}}\cos^2\sqrt{x}} \times \lim_{h\to 0} \frac{x+h-x}{h\left(\sqrt{x+h}+\sqrt{x}\right)} \qquad \left(\text{using } \lim_{t\to 0}\frac{\sin t}{t} = 1\right)$$

$$\Rightarrow \quad f'(x) = \frac{1}{2\sqrt{\tan\sqrt{x}}\cos^2\sqrt{x}} \; \frac{1}{2\sqrt{x}}$$

Example : 18

If y = f ($\sin^2$x) and $f'(x) = \frac{1+x}{1-x}$, then show that $\frac{dy}{dx}$ = 2 tan x (1 + $\sin^2$x)

Solution

Let u $\sin^2$x

Using chain rule : $\frac{dy}{dx} = f'(u)\,\frac{du}{dx}$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{1+u}{1-u}\,\frac{d}{dx}(\sin^2 x) = \frac{1+\sin^2 x}{1-\sin^2 x}(2\sin x\cos x) = 2\tan x\,(1+\sin^2 x)$$

Example : 19

A function f : R $\to$ R satisfy the equation f(x + y) = f(x) f(y) for all x, y in R and f(x) $\neq$ 0 for any x in R. Let the function be differentiable at x = 0 and f′(0) = 2. Show that f′(x) = 2f(x) for all x in R. Hence determine f(x).

Solution

In f(x + y) = f(x) f(y) substitute y = 0

$\Rightarrow$ f(x + 0) = f(x) f(0)

$\Rightarrow$ f(x) = f(x) f(0)

$\Rightarrow$ f(0) = 1 ($\because$ f(x) $\neq$ 0) ....(i)

Consider $f'(0) = \lim_{h\to 0}\frac{f(0+h)-f(0)}{h}$

$$\Rightarrow \quad 2 = \lim_{h\to 0}\frac{f(h)-1}{h} \qquad \text{............(ii)}$$

Consider $f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{f(x)f(h)-f(x)}{h} = f(x)\lim_{h\to 0}\frac{f(h)-1}{h}$

= f(x) (2) [using (2)]

$\Rightarrow$ f′(x) = 2f(x)

$$\Rightarrow \quad \frac{f'(x)}{f(x)} = 2$$

$$\Rightarrow \quad \frac{d}{dx}[\log f(x)] = \frac{d}{dx}(2x)$$

$\Rightarrow$ log f(x) = 2x $\Rightarrow$ f(x) = e^{2x}

Example : 20

(Logarithmic differentiation) Find dy/dx for the functions.

(i) $y = \left(1+\frac{1}{x}\right)^x + x^{1+\frac{1}{x}}$ (ii) $y = \frac{(2x+1)^3\sqrt{1-x^2}}{(3x-2)^2\ 2^x}$ (iii) $y = \log_x (\log x)$

Solution

(i) Let $u = \left(1+\frac{1}{x}\right)^x$ and $v = x^{1+\frac{1}{x}}$

$\Rightarrow \quad y = u + v$

$\Rightarrow \quad \frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx}$ ....(i)

(ii) $y = \frac{(2x+1)^3\sqrt{1-x^2}}{(3x-2)^2\ 2^x}$

Now $u = \left(1+\frac{1}{x}\right)^x$

$\Rightarrow \quad \log u = x \log \left(1+\frac{1}{x}\right) = x \log (x + 1) - x \log x$

$\Rightarrow \quad \frac{1}{u}\frac{du}{dx} = \frac{x}{x+1} + \log (x + 1) - \left(\frac{x}{x} + \log x\right)$

$\Rightarrow \quad \frac{du}{dx} = u \left(\log\frac{x+1}{x} - \frac{1}{x+1}\right)$ ....(ii)

consider $v = x^{1+\frac{1}{x}}$

$\Rightarrow \quad \log v = \left(1+\frac{1}{x}\right) \log x$

$\Rightarrow \quad \frac{1}{v}\frac{dv}{dx} = \left(1+\frac{1}{x}\right)\frac{1}{x} + \log x \left(-\frac{1}{x^2}\right)$

$\Rightarrow \quad \frac{dv}{dx} = \frac{v}{x^2}(x + 1 - \log x)$ ....(iii)

Substituting from (ii) and (iii) into (i)

$$\frac{dy}{dx} = \left(1+\frac{1}{x}\right)^x \left(\log\frac{x+1}{x} - \frac{1}{x+1}\right) + \frac{x^{1+\frac{1}{x}}}{x^2} \times (x + 1 - \log x)$$

(ii) Taking log on both sides :

$\log y = 3 \log (2x + 1) + 1/2 \log (1 - x^2) - 2 \log (3x - 2) - x \log 2$

Differentiating with respect to x, $\frac{1}{y}\frac{dy}{dx} = \frac{3(2)}{2x+1} + \frac{-2x}{2(1-x^2)} = \frac{2(3)}{3x-2} - \log 2$

$$\frac{dy}{dx} = \frac{(2x+1)^3\sqrt{1-x^2}}{(3x-2)^2 2^x} \times \left[\frac{6}{2x+1} - \frac{x}{1-x^2} - \frac{6}{3x-2} - \log 2\right]$$

(iii) $y = \log_x (\log x)$

$$y = \frac{\log \log x}{\log x} \qquad \left(\text{using } \log_a b = \frac{\log_m b}{\log_m a}\right)$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{\log x\left(\frac{1}{\log x}\frac{1}{x}\right) - (\log \log x)\frac{1}{x}}{(\log x)^2}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{1}{x(\log x)^2} \ (1 - \log \log x)$$

Example : 21

(Implicit function) Find the expression for $\frac{dy}{dx}$ for the following implicit function.

(a) $x^{\sin y} = y^{\sin x}$ (b) $x^3 + y^3 - 3xy = 1$

Solution

(a) $x^{\sin y} = y^{\sin x}$

$\Rightarrow \quad \sin y \log x = \sin x \log y$

Differentiating with respect to x :

$$\sin y \ \frac{1}{x} + \log x \cos y \ \frac{dy}{dx} = \sin x \ \frac{1}{y} \ \frac{dy}{dx} + \log y \cos x$$

$$\Rightarrow \quad \frac{dy}{dx}\left(\log x \cos y - \frac{\sin x}{y}\right) = \cos x \log y - \frac{\sin y}{x}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{xy \cos x \log y - y \sin y}{xy \log x \cos y - x \sin x} \qquad \frac{1+x^2}{1-x^2}$$

(b) $x^3 + y^3 - 3xy = 1$

Differentiating with respect to x;

$$3x^2 + 3y^2 \ \frac{dy}{dx} - 3\left[x\frac{dy}{dx} + y, 1\right] = 0$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{y - x^2}{y^2 - x}$$

Example : 22

(Inverse circular functions) Find $\frac{dy}{dx}$ if

(1) $y = \tan^{-1}\left(\frac{a\cos - b\sin x}{b\cos x + a\sin x}\right)$ (2) $y = \tan^{-1}\left(\frac{x}{1+\sqrt{1-x^2}}\right)$

(2) $y = \tan^{-1}\frac{4x}{1+5x^2} + \tan^{-1}\frac{2+3x}{3-2x}$ (4) $y = \sin^{-1}\frac{2x}{1+x^2} + \sec^{-1}$

Solution

(1) $y = \tan^{-1}\left(\frac{a\cos x - b\sin x}{b\cos x + a\sin x}\right) \quad \Rightarrow \quad y = \tan^{-1}\left(\frac{a/b - \tan x}{1 + a/b \tan x}\right)$

$= \tan^{-1}(a/b) - \tan^{-1}\tan x = \tan^{-1}(a/b) - x$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{d}{dx}\left(\tan^{-1}\frac{a}{b} - x\right) = 0 - 1 = -1$$

(2) $y = \tan^{-1}\left(\frac{x}{1+\sqrt{1-x^2}}\right)$

Substitute $x = \sin\theta$ ....(i)

$y = \tan^{-1}\left(\frac{\sin\theta}{1+\sqrt{1-\sin^2\theta}}\right)$

$y = \tan^{-1}\left(\frac{\sin\theta}{1+\cos\theta}\right)$

$y = \tan^{-1}\left(\frac{2\sin\theta/2\cos\theta/2}{2\cos^2\theta/2}\right)$

$y = \tan^{-1}\tan\theta/2 = \frac{\theta}{2}$

using (i), $y = \frac{1}{2}\sin^{-1}x$

$\Rightarrow \quad \frac{dy}{dx} = \frac{1}{2\sqrt{1-x^2}}$

(3) $y = \tan^{-1}\frac{4x}{1+5x^2} + \tan^{-1}\frac{2+3x}{3-2x}$

$y = \tan^{-1}\frac{5x-x}{1+5x\,x} + \tan^{-1}\left(\frac{\frac{2}{3}+x}{1-\frac{2}{3}x}\right)$ $\quad \frac{1+x^2}{1-x^2}$

$y = \tan^{-1}5x - \tan^{-1}x + \tan^{-1}2/3 + \tan^{-1}x$

$y = \tan^{-1}5x + \tan^{-1}2/3$

$\Rightarrow \quad \frac{dy}{dx} = \frac{5}{1+25x^2}$

(4) $y = \sin^{-1}\frac{2x}{1+x^2} + \sec^{-1}$

Substitute $x = \tan\theta$ ....(i)

$y = \sin^{-1}\left(\frac{2\tan\theta}{1+\tan^2\theta}\right) + \sec^{-1}\left(\frac{1+\tan^2\theta}{1-\tan^2\theta}\right)$

$y = \sin^{-1}\sin 2\theta + \cos^{-1}\cos 2\theta$

$y = 2\theta + 2\theta$

$y = 4\theta = 4\tan^{-1}x$ (using (i))

$\Rightarrow \quad \frac{dy}{dx} = \frac{4}{1+x^2}$

Example : 23

Show that = 1 if $y = \cos^{-1}\left(\frac{\cos x + 4\sin x}{\sqrt{17}}\right)$

Solution

We can write

$$\cos x + 4 \sin x = \sqrt{17}\left[\frac{1}{\sqrt{17}}\cos x + \frac{4}{\sqrt{17}}\sin x\right] = \sqrt{17}\cos(x - \tan^{-1}4)$$

Hence $y = \cos^{-1}\left(\frac{\sqrt{17}\cos(x-\tan^{-1}4)}{\sqrt{17}}\right) \Rightarrow y = x - \tan^{-1} 4$

$\Rightarrow \frac{dy}{dx} = 1$

Example : 24

If $\sqrt{1-x^2} + \sqrt{1-y^2} = a(x - y)$, Show that $\frac{dy}{dx} = \sqrt{\frac{1-y^2}{1-x^2}}$

Solution

Substitute $x = \sin\alpha$ and $y = \sin\beta$ ....(i)

$\Rightarrow \sqrt{1-\sin^2\alpha} + \sqrt{1-\sin^2\beta} = a(\sin\alpha - \sin\beta)$

$\Rightarrow \cos\alpha + \cos\beta = a(\sin\alpha - \sin\beta)$

$\Rightarrow \frac{2\cos\left(\frac{\alpha+\beta}{2}\right)\cos\frac{\alpha-\beta}{2}}{2\cos\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right)} = a$ $\frac{dy}{dx}$

$\Rightarrow \cot\left(\frac{\alpha-\beta}{2}\right) = a$

$\Rightarrow \alpha - \beta = 2\cot^{-1} a$

$\Rightarrow \sin^{-1} x - \sin^{-1} y = 2\cot^{-1} a,$ [using (i)]

differentiating with respect to x.

$$\frac{1}{\sqrt{1-x^2}} - \frac{1}{\sqrt{1-y^2}}\frac{dy}{dx} = 0$$

$\Rightarrow \frac{dy}{dx} = \sqrt{\frac{1-y^2}{1-x^2}}$

Example : 25

If $x = a(\cos t + \log\tan t/2)$, $y = a\sin t$ find d^2y/dx^2 at $t = \pi/4$

Solution

$$\frac{dy}{dt} = a\cos t$$

$$\frac{dx}{dt} = a\left(-\sin t + \frac{1}{\tan t/2}\frac{\sec^2 t/2}{2}\right) = a\left(-\sin t + \frac{1}{\sin t}\right)$$

$\Rightarrow \frac{dx}{dt} = \frac{a\cos^2 t}{\sin t}$

$\therefore \quad \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{a\cos t}{a\cos^2 t/\sin t} \quad \Rightarrow \quad \frac{dy}{dx} = \tan t$

Now $\quad \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{\frac{d}{dt}(dy/dx)}{\frac{dx}{dt}}$

$$\frac{d^2y}{dx^2} = \frac{\sec^2 t}{a\cos^2 t/\sin t} = \frac{\sin t}{a\cos^4 t}$$

$$\left.\frac{d^2y}{dx^2}\right]_{t=\pi/4} = \frac{\sin\pi/4}{a\cos^4\pi/4} = \frac{2\sqrt{2}}{a}$$

Example : 25

If x = a (cos t + log tan t/2), y = a sin t find d^2y/dx^2 at t = π/4.

Solution

$$\frac{dy}{dt} = a\cos t$$

$$\frac{dx}{dt} = a\left(-\sin t + \frac{1}{\tan t/2}\frac{\sec^2 t/2}{2}\right) = a\left(-\sin t + \frac{1}{\sin t}\right)$$

$\Rightarrow \quad \frac{dx}{dt} = \frac{a\cos^2 t}{\sin t}$

$\therefore \quad \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{a\cos t}{a\cos^2 t/\sin t} \quad \Rightarrow \quad \frac{dy}{dx} = \tan t$

Now $\quad \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{\frac{d}{dt}(dy/dx)}{\frac{dx}{dt}}$

$$\frac{d^2y}{dx^2} = \frac{\sec^2 t}{a\cos^2 t/\sin t} = \frac{\sin t}{a\cos^4 t}$$

$$\left.\frac{d^2y}{dx^2}\right]_{t=\pi/4} = \frac{\sin\pi/4}{a\cos^4\pi/4} = \frac{2\sqrt{2}}{a}$$

Example : 26

If $x\sqrt{1+y} + y\sqrt{1+x} = 0$, then shown that $\frac{dy}{dx} = \frac{-1}{(1+x)^2}$

Solution

$x\sqrt{1+y} = -y\sqrt{1+x}$

Squaring, we get :

$x^2(1 + y) = y^2(1 + x)$

$\Rightarrow \quad x^2 + x^2y - y^2 - xy^2 = 0$

$\Rightarrow \quad (x^2 - y^2) + xy(x - y) = 0$

$\Rightarrow \quad (x - y)(x + y + xy) = 0$

$\Rightarrow \quad y = x$ or $x + y + xy = 0$

Since y = x does not satisfy the give function, we reject it.

$\therefore$ $x + y + xy = 0$

$\Rightarrow$ $y = \dfrac{-x}{1+x}$

$\Rightarrow$ $\dfrac{dy}{dx} = -\dfrac{(1+x)-x,1}{(1+x)^2} = \dfrac{-1}{(1+x)^2}$

Example : 27

If $y = \dfrac{\sqrt{a^2+x^2}+\sqrt{a^2-x^2}}{\sqrt{a^2+x^2}-\sqrt{a^2-x^2}}$, then show that $\dfrac{dy}{dx} = -\dfrac{2a^2}{x^3}\left(1+\dfrac{a^2}{\sqrt{a^4-x^4}}\right)$

Solution

On rationalising the denominator, we get :

$$y = \frac{\left(\sqrt{a^2+x^2}+\sqrt{a^2-x^2}\right)^2}{2x^2}$$

$$y = \frac{2a^2+2\sqrt{a^4-x^4}}{2x^2}$$

$$y = \frac{a^2}{x^2} + \frac{\sqrt{a^4-x^4}}{x^2}$$

$\Rightarrow$ $\dfrac{dy}{dx} = \dfrac{-2a^2}{x^3} + \dfrac{x^2\dfrac{-4x^3}{2\sqrt{a^4-x^4}} - \sqrt{a^4-x^4}\,(2x)}{x^4}$

$\Rightarrow$ $\dfrac{dy}{dx} = \dfrac{-2a^2}{x^3} + \dfrac{-2x^4-2(a^4-x^4)}{x^3\sqrt{a^4-x^4}}$

$\Rightarrow$ $\dfrac{dy}{dx} = \dfrac{-2a^2}{x^3}\left[1+\dfrac{a^2}{\sqrt{a^4-x^4}}\right]$

Example : 28

If $y = a^{x^{a^{x^{\cdots\cdots\infty}}}}$, then find $\dfrac{dy}{dx}$

Solution

$y = a^{x^{a^{x^{\cdots\cdots\infty}}}}$ can be written as $y = a^{x^y}$

$\Rightarrow$ $\log y = x^y \log a$

$\Rightarrow$ $\log\log y = y \log x + \log\log a$

differentiating with respect to x;

$\dfrac{1}{\log y}\,\dfrac{1}{y}\,\dfrac{dy}{dx} = y\,.\,\dfrac{1}{x} + \log x\,\dfrac{dy}{dx}$

$\Rightarrow$ $\dfrac{dy}{dx}\left(\dfrac{1}{y\log y} - \log x\right) = \dfrac{y}{x}$

$\Rightarrow$ $\dfrac{dy}{dx} = \dfrac{y^2\log y}{x(1-y\log x\log y)}$

Example : 29

If $y = \cos^{-1}\dfrac{a+b\cos x}{b+a\cos x}$, b > a, then show that $\dfrac{dy}{dx} = \dfrac{\sqrt{b^2-a^2}}{b+a\cos x}$

Solution

Differentiating y with respect to x

$$\frac{dy}{dx} = \frac{-1}{\sqrt{1-\left(\frac{a+b\cos x}{b+a\cos x}\right)^2}} \times \frac{(b+a\cos x)(-b\sin x)-(a+b\cos x)(-a\sin x)}{(b+a\cos x)^2}$$

$$= \frac{-(b+a\cos x)}{\sqrt{(b^2-a^2)-(b^2-a^2)\cos^2 x}} \; \frac{-b^2\sin x+a^2\sin x}{(b+a\cos x)^2} = \frac{(b^2-a^2)\sin x}{\sqrt{b^2-a^2}\sqrt{1-\cos^2 x}(b+a\cos x)}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{\sqrt{b^2-a^2}}{b+a\cos x}$$

Example : 30

If sin y = x sin (a + y), then show that :

(i) $\dfrac{dy}{dx} = \dfrac{\sin^2(a+y)}{\sin a}$ (ii) $\dfrac{dy}{dx} = \dfrac{\sin a}{1+x^2-2x\cos x}$

Solution

(i) As $\dfrac{dy}{dx}$ should not contain x, we write $\dfrac{\sin y}{\sin(a+y)}$ = x and differentiating with respect to x;

$$\left[\frac{\sin(a+y)\cos y-\sin y\cos(a+y)}{\sin^2(a+y)}\right]\frac{dy}{dx} = 1$$

$$\Rightarrow \quad \frac{\sin(a+y-y)}{\sin^2(a+y)}\frac{dy}{dx} = 1 \quad \Rightarrow \quad \frac{dy}{dx} = \frac{\sin^2(a+y)}{\sin a}$$

(ii) As $\dfrac{dy}{dx}$ should not contain y, we try to express y explicitly in terms of x.

sin y = x (sin a cos y + cos a sin y)

$$\Rightarrow \quad \tan y = \frac{x\sin a}{1-x\cos a} \qquad \Rightarrow \quad y = \tan^{-1}\left(\frac{x\sin a}{1-x\cos a}\right)$$

Now differentiate with respect x;

$$\frac{dy}{dx} = \frac{1}{1+\frac{x^2\sin^2 a}{(1-x\cos a)^2}} \; \frac{(1-x\cos a)\sin a - x\sin a(-\cos a)}{(1-x\cos a)^2} = \frac{\sin a}{(1-x\cos a)^2+x^2\sin^2 a}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{\sin a}{1+x^2-2x\cos a}$$

Example : 31

If $y = e^{mx}(ax + b)$, where a, b, m are constants, show that : $\frac{d^2y}{dx^2} - 2m\frac{dy}{dx} + m^2y = 0$

Solution

$y = e^{mx}(ax + b)$ ....(i)

$\frac{dy}{dx} = (a)\,e^{mx} + m\,(ax + b)\,e^{mx}$

using (i), $\frac{dy}{dx} = a\,e^{mx} + my$ ....(ii)

Again differentiating with respect to x;

$\frac{d^2y}{dx^2} = ame^{mx} + m\frac{dy}{dx}$

Substituting for $a\,e^{mx}$ from (ii), we get

$\frac{d^2y}{dx^2} = m\left(\frac{dy}{dx} - my\right) + m\frac{dy}{dx}$

$\Rightarrow \quad \frac{d^2y}{dx^2} - 2m\frac{dy}{dx} + m^2y = 0$

Example : 32

If $y = x\log\frac{x}{a+bx}$, the show that : $x^3\frac{d^2y}{ax^2} = \left(x\frac{dy}{dx} - y\right)^2$

Solution

$y = x\log x - x\log(a + bx)$ ....(i)

$\Rightarrow \quad \frac{dy}{dx} = x\,\frac{1}{x} + \log x - \frac{xb}{a+bx} - \log(a + bx)$

$\Rightarrow \quad \frac{dy}{dx} = [\log x - \log(a + bx)] + \frac{a}{a+bx}$

$\Rightarrow \quad \frac{dy}{dx} = \frac{y}{x} + \frac{a}{a+bx}$ [using (i)]

$x\frac{dy}{dx} - y = \frac{ax}{a+bx}$ ....(ii)

Again differentiating with respect to x, we get ;

$\left(x\frac{d^2y}{dx^2} + \frac{dy}{dx}\right) - \frac{dy}{dx} = \frac{(a+bx)a - ax.b}{(a+bx)^2}$

$\Rightarrow \quad x\frac{d^2y}{dx^2} = \frac{a^2}{(a+bx)^2}$

$\Rightarrow \quad x^3\frac{d^2y}{dx^2} = \frac{a^2x^2}{(a+bx)^2}$

$\Rightarrow \quad x^3\frac{d^2y}{dx^2} = \left(x\frac{dy}{dx} - y\right)^2$ [using (ii) in R,H.S)

Example : 33

If x = sin t and y = cos pt, show that : $(1 - x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} + p^2y = 0$

Solution

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{-p\sin pt}{\cos t}$$

As the equation to be derived does no contain t, we eliminate t using expressions for x and y.

$$\frac{dy}{dx} = \frac{-p\sqrt{1-y^2}}{\sqrt{1-x^2}}$$

$$\Rightarrow \quad \sqrt{1-x^2}\frac{dy}{dx} = -p\sqrt{1-y^2}$$

As the equation to be derived does not contain any square root, we square and then differentiate.

$$(1 - x^2)\left(\frac{dy}{dx}\right)^2 = p^2(1 - y^2)$$

$$(1 - x^2)\, 2\frac{dy}{dx}\frac{d^2y}{dx^2} + (-2x)\left(\frac{dy}{dx}\right)^2 = p^2\left(-2y\frac{dy}{dx}\right)$$

$$\Rightarrow \quad (1 - x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} = -p^2y$$

$$\Rightarrow \quad (1 - x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} + p^2y = 0$$

Example : 34

If $x = at^3$, $y = bt^2$ (t a parameter), find

(i) $\frac{d^3y}{dx^3}$ (ii) $\frac{d^3x}{dy^3}$

Solution

(i) $x = at^3 \Rightarrow \frac{dx}{dt} = 3at^2$

$y = bt^2 \Rightarrow \frac{dy}{dt} = 2bt$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{2bt}{3at^2} = \frac{2b}{3at}$$

$$\Rightarrow \quad \frac{d^2y}{dx^2} = \frac{2b}{3a}\frac{d}{dx}\left(\frac{1}{t}\right) = \frac{2b}{3a}, \frac{-1}{t^2}\cdot\frac{dt}{dx} = \frac{-2b}{3at^2}\cdot\frac{1}{3at^2} = \frac{-2b}{9a^2t^4}$$

Again differentiating both sides w.r.t. x,

$$\frac{d^3y}{dx^3} = \frac{d}{dt}\left(\frac{d^2y}{dx^2}\right)\frac{dt}{dx} = -\frac{2b}{9a^2}\frac{d}{dt}\left(\frac{1}{t^4}\right)\frac{dt}{dx} = \frac{-2b}{9a^2}\cdot\frac{-4}{t^5}\cdot\frac{1}{3at^2} = \frac{8b}{27a^3t^7}$$

(ii) $x = at^3$, $y = bt^2$

$\frac{dx}{dt} = 3a^2$; $\frac{dy}{dt} = 2bt$

$$\Rightarrow \quad \frac{dx}{dt} = \frac{dx/dt}{dy/dt} = \frac{3at^2}{2bt} = \frac{3at}{2b}$$

$$\Rightarrow \quad \frac{d^2x}{dy^2} = \frac{3a/2b}{dy/dt} = \frac{3a}{2b} \cdot \frac{1}{2bt} = \frac{3a}{4b^2t}$$

$$\Rightarrow \quad \frac{d^3x}{dy^3} = \frac{d}{dy}\left(\frac{3a}{4b^2t}\right) = \frac{3a}{4b^2} \cdot \frac{d}{dt}\left(\frac{1}{t}\right) \cdot \frac{1}{dy/dt} = \frac{3a}{4b^2}\left(-\frac{1}{t^2}\right)\left(\frac{1}{2bt}\right) = \frac{-3a}{8b^3t^3}$$

Example : 35

If y = f(x), express $\frac{d^2x}{dy^2}$ in terms of $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$.

Solution

$$\frac{dx}{dy} = \frac{1}{\frac{dy}{dx}} \qquad \left(\frac{dy}{dx} \neq 0\right)$$

$$\Rightarrow \quad \frac{d^2x}{dy^2} = \frac{d}{dy}\left(\frac{1}{dy/dx}\right) = \frac{d}{dx}\left(\frac{1}{dy/dx}\right) \cdot \frac{dx}{dy} = -\frac{1}{\left(\frac{dy}{dx}\right)^2} \frac{d}{dx}\left(\frac{dy}{dx}\right) \cdot \frac{dx}{dy}$$

$$= -\frac{\frac{d^2y}{dx^2}}{\left(\frac{dy}{dx}\right)^3} \qquad \because \quad \frac{dx}{dy} = \frac{1}{\frac{dy}{dx}}$$

Example : 36

Change the independent variable x to θ in the equation $\frac{d^2y}{dx^2} + \frac{2x}{1+x^2} \cdot \frac{dy}{dx} + \frac{y}{(1+x^2)^2} = 0$, by means of the transformation x = tan θ

Solution

$$x = \tan\theta \quad \Rightarrow \quad \frac{dx}{d\theta} = \sec^2\theta$$

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \cos^2\theta \cdot \frac{dy}{d\theta} \quad \Rightarrow \quad \frac{d^2y}{dx^2} = -2\cos\theta \cdot \sin\theta \cdot \frac{d\theta}{dx} \cdot \frac{dy}{d\theta} + \cos^2\theta \cdot \frac{d^2y}{d\theta^2} \cdot \frac{d\theta}{dx}$$

$$= -2\cos\theta \cdot \sin\theta \cdot \cos^2\theta \cdot \frac{dy}{d\theta} + \cos^2\theta \cdot \frac{d^2y}{d\theta^2} \cdot \cos^2\theta$$

$$= -2\sin\theta \cdot \cos^3\theta \cdot \frac{dy}{d\theta} + \cos^4\theta \cdot \frac{d^2y}{d\theta^2}$$

Putting the values of x, $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ in the given equation, $\frac{d^2y}{dx^2} + \frac{2x}{1+x^2} \frac{dy}{dx} + \frac{y}{(1+x^2)^2} = 0$

we get $-2\sin\theta\cos^3\theta \frac{dy}{d\theta} + \cos^4\theta \frac{d^2y}{d\theta^2} + \frac{2\tan\theta}{1+\tan^2\theta}\cos^2\theta \frac{dy}{d\theta} + \frac{y}{(1+\tan^2\theta)^2} = 0$

$$-2\sin\theta\cos^3\theta \frac{dy}{d\theta} + \cos^4\theta \frac{d^2y}{d\theta^2} + 2\sin\theta\cos^3\theta \frac{dy}{d\theta} + y\cos^4\theta = 0$$

i.e. $\frac{d^2y}{d\theta^2} + y = 0$

Example : 37

Differentiate x^x (x > 0) from first principles.

Solution

Let $f(x) = x^x = e^{x \ell nx}$

From first principles,

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

$$f'(x) = \lim_{h\to 0} \frac{e^{(x+h)\,\ell n(x+h)} - e^{x\ell nx}}{h} = \lim_{h\to 0} \frac{e^{x\ell nx}\left[e^{(x+h)\ell n(x+h)-x\ell nx}-1\right]}{h}$$

$$= \lim_{h\to 0} \frac{e^{x\ell n\,x}\left[e^{(x+h)\ell n(x+h)-x\ell nx}-1\right]}{(x+h)\ell n\,(x+h)-x\ell nx} \cdot \lim_{h\to 0} \frac{(x+h)\ell n(x+h)-x\ell nx}{h}$$

$$= e^{x\ell n\,x} \cdot \lim_{h\to 0} \frac{(x+h)\ell n(x+h)-x\ell n(x+h)+x\ell n(x+h)-x\ell nx}{h} \qquad \left[\text{using} : \lim_{h\to 0} \frac{e^h-1}{h} = 1\right]$$

$$= e^{x\ell n\,x} \cdot \left[\lim_{h\to 0} \frac{\ell n(x+h)[x+h-x]}{h} + \lim_{h\to 0} \frac{x\ell n(x+h)-\ell nx}{h}\right]$$

$$= e^{x\ell n\,x} \cdot \left[\lim_{h\to 0} \ell n(x+h) + \lim_{h\to 0} \frac{x\left[\ell n\left(\frac{x+h}{x}\right)\right]}{h}\right] = e^{x\ell n\,x} \cdot \left[\ell n\,x + \lim_{h\to 0}\left[\ell n\left(1+\frac{h}{x}\right)^{\frac{x}{h}}\right]\right]$$

$$= ex\,\ell n\,x \,.\, (\ell n\,x + 1) \qquad \left[\text{using} : \lim_{t\to 0} \ell n\,(1+t)^{\frac{1}{t}} = \ell n\,e = 1\right]$$

$\Rightarrow \quad f'(x) = x^x (1 + \ell n\,x)$

Example : 38

If $y = \log_u |\cos 4x| + |\sin x|$, where $u = \sec 2x$, find dy/dx at $x = -\pi/6$

Solution

In the sufficiently closed neighbourhood of $-\pi/6$ both cos 4x and sin x are negative. So for differentiating y, we can take $|\cos 4x| = -\cos 4x$ and $|\sin x| = -\sin x$.

Thus

$y = \log_u (-\cos 4x) + (-\sin x) = \log_{\sec 2x} (-\cos 4x) + (-\sin x)$

$$= \frac{\log(-\cos 4x)}{\log \sec 2x} - \sin x$$

On differentiating wit respect to x, we get

$$\frac{dy}{dx} = \frac{\frac{(4\sin 4x) x \log \sec 2x}{-\cos 4x} - \log(-\cos 4x)\frac{\sec 2x \times \tan 2x}{\sec 2x} \times 2}{(\log \sec 2x)^2} - \cos x$$

$$= \frac{-4\tan 4x \times \log\sec 2x - 2\tan 2x \times \log(-\cos 4x(}{(\log\sec 2x)^2} - \cos x$$

Taking derivative at $x = -\pi/6$, we get

$$\left[\frac{dy}{dx}\right]_{x=-\pi/6} = \frac{-4\tan(-2\pi/3)\times\log\sec\pi/3 - 2\tan(-\pi/3)\times\log(-\cos(-2\pi/3))}{(\log 2)^2} - \frac{\sqrt{3}}{2} = \frac{\sqrt{3}}{2} - \frac{6\sqrt{3}}{\log 2}$$

Example : 39

Test the differentiability of the following function at x = 0.

$$f(x) = \begin{cases} e^{-1/x^2}\sin\left(\frac{1}{x}\right) & ; \quad x \neq 0 \\ 0 & ; \quad x = 0 \end{cases}$$

Solution

Checking differentiability at x = 0

$$\text{Right hand derivative} = \lim_{h\to 0} \frac{f(0+h)-f(0)}{h} = \lim_{h\to 0} \frac{e^{\frac{-1}{(0+h)^2}}\sin\left(\frac{1}{0+h}\right)-0}{h}$$

$$= \lim_{h\to 0} \frac{\sin\left(\frac{1}{h}\right)}{he^{\frac{1}{h^2}}} = \lim_{h\to 0} \frac{\sin\left(\frac{1}{h}\right)}{h\left(1+\frac{1}{h^2}+\frac{1}{h^4 2!}+....\right)}$$

$$= \lim_{h\to 0} \frac{\sin\left(\frac{1}{h}\right)}{\left(h+\frac{1}{h}+\frac{1}{h^3 2!}+....\right)} = \frac{\text{a finite quantity}}{0+\infty} = 0$$

(because sin (1/h) is finite and oscillates between – 1 to + 1).

$$\text{Left Hand Derivative} = \lim_{h\to 0} \frac{f(0-h)-f(0)}{-h} = \lim_{h\to 0} \frac{e^{\frac{-1}{(\theta-h)^2}}\sin\left(\frac{1}{\theta-h}\right)-0}{-h}$$

$$= \lim_{h\to 0} \frac{\sin\left(\frac{1}{h}\right)}{he^{\frac{1}{h^2}}} = \lim_{h\to 0} \frac{\sin\left(\frac{1}{h}\right)}{h\left(1+\frac{1}{h^2}+\frac{1}{h^4 2!}+....\right)} = \frac{\text{a finite quantity}}{0+\infty} = 0$$

(because sin (1/h) is finite and oscillates between – 1 to + 1)

As Left Hand Derivative = Right Hand Derivative, the function f(x) is differentiable at x = 0.

Example : 40

The function f is defined by y = f(x) where x = 2t – |t|, $y = t^2 + t|t|$, $t \in R$. Draw the graph of f(x) for the interval $-1 \le x \le 1$. Also discuss the continuity and differentiability at x = 0.

Solution

It is given that : x = x = 2t – |t| and $y = t^2 + t|t|$.

Consider $t \ge 0$ x = 2t – t = t ....(i)

and $y = t^2 + t \times t = 2t^2$ ....(ii)

Eliminating t from (i) and (ii), we get $y = 2x^2$

So $y = 2x^2$ for $x > 0$ (because $t \ge 0 \Rightarrow x \ge 0$)

Consider $t < 0$ x = 2t + t = 3t ....(iii)

and $y = t^2 + t \times (-t) = 0$ ....(iv)

Eliminating t from (iii) and (iv), we get y = 0

So y = 0 for x < 0 (because $t < 0 \Rightarrow x < 0$)

In the closed interval $-1 \le x \le 1$, the function f(x) is :

$$f(x) = \begin{cases} 2x^2 & ; \quad x \ge 0 \\ 0 & ; \quad x < 0 \end{cases}$$

Checking differentiability at x = 0

$$LHD = \lim_{h\to 0} \frac{f(0-h)-f(0)}{-h} = \lim_{h\to 0} \frac{0-0}{-h} = 0$$

$$RHD = \lim_{h\to 0} \frac{f(0+h)-f(0)}{h} = \lim_{h\to 0} \frac{2(0+h)^2-0}{h} = \lim_{h\to 0} 2h = 0$$

As LHD = RHD, f(x) is continuous and differentiable at x = 0 (because if function is differential, it must be continuous)

Example : 41

If x < 1, prove that : $\frac{1-2x}{1-x+x^2} + \frac{2x-4x^3}{1-x^2+x^4} + \frac{4x^3-8x^7}{1-x^4+x^8} + \ldots\ldots \infty = \frac{1+2x}{1+x+x^2}$

Solution

$(1 + x + x^2)(1 - x + x^2) = (1 + x^2)^2 - x^2 = 1 + x^2 + x^4$

$(1 + x + x^2)(1 - x + x^2)(1 - x^2 + x^4) = (1 + x^2 + x^4)(1 - x^2 + x^4) = (1 + x^4)^2 - x^4 = 1 + x^4 + x^8$

Continuing the same way, we can obtain :

$$(1 + x + x^2)(1 - x + x^2)(1 - x^2 + x^4) \ldots\ldots\ldots\ldots \left(1 - x^{2^{n-1}} + x^{2^n}\right) = 1 + x^{2^n} +$$

Taking limit $n \to \infty$, we get

$(1 + x + x^2)(1 - x + x^2)(1 - x^2 + x^4) \ldots\ldots\ldots\ldots = 1 \quad (\because x < 1)$

Take log of both sides to get

$\log(1 + x + x^2) + \log(1 - x + x^2) + \log(1 - x^2 + x^4) + \ldots\ldots\ldots\ldots = 0$

Differentiate both sides with respect to x :

$$\frac{1+2x}{1+x+x^2} + \frac{-1+2x}{1-x+x^2} + \frac{-2x+4x^3}{1-x^2+x^4} + \ldots\ldots = 0$$

$$\Rightarrow \quad \frac{1-2x}{1-x+x^2} + \frac{2x-4x^3}{1-x^2+x^4} + \ldots\ldots = \infty \ \frac{1+2x}{1+x+x^2} \quad x^{2^{n+1}}$$

Hence proved

Example : 42

Find the derivative with respect to x of the function:

$$(\log_{\cos x} \sin x)(\log_{\sin x} \cos x)^{-1} + \sin^{-1}\left(\frac{2x}{1+x^2}\right) \text{ at } x = \pi/4.$$

Solution

Let $y = (\log_{\cos x} \sin x)(\log_{\sin x} \cos x)^{-1} + \sin^{-1}\left(\frac{2x}{1+x^2}\right)$, $u = (\log_{\cos x} \sin x)(\log_{\sin x} \cos x)^{-1}$ and $v = \sin^{-1}\left(\frac{2x}{1+x^2}\right)$

$\Rightarrow \quad y = u + v \quad \ldots\ldots\ldots\ldots(i)$

consider u

$u = (\log_{\cos x} \sin x)(\log_{\sin x} \cos x)^{-1} = (\log_{\cos x} \sin x)(\log_{\cos x} \sin x) = (\log_{\cos x} \sin x)^2$

$$\frac{du}{dx} = \frac{d}{dx}(\log_{\cos x} \sin x)^2 = \frac{d}{dx}\left(\frac{\log_e \sin x}{\log_e \cos x}\right)^2 = 2\left(\frac{\log_e \sin x}{\log_e \cos x}\right)\frac{d}{dx}\left(\frac{\log_e \sin x}{\log_e \cos x}\right)$$

$$= 2(\log_{\cos x} \sin x) \times \left(\frac{\log_e \cos x \frac{\cos x}{\sin x} - \log_e \sin x \frac{-\sin x}{\cos x}}{(\log_e \cos x)^2}\right)$$

$$= 2(\log_{\cos x} \sin x) \times \left(\frac{\cot x \log_e \cos x + \tan x \log_e \sin x}{(\log_e \cos x)^2}\right)$$

consider y

$v = \sin^{-1}\left(\frac{2x}{1+x^2}\right)$

put x = tan θ ⇒ $v = \sin^{-1}(\sin 2\theta) = 2\theta = 2\tan^{-1}x$

[for $-\pi/2 \le 2\theta \le \pi/2$ ⇒ $-\pi/4 \le \theta \le \pi/4 \le \theta \le \pi/4$ ⇒ $-1 \le x \le 1$

⇒ we can use this definition for x = π/4]

$$\Rightarrow \quad \frac{dv}{dx} = 2\frac{d}{dx}\tan^{-1}x = \frac{2}{1+x^2}$$

Differentiating (i) with respect to x at x = π/4, we get

$$= \left[\frac{du}{dx}\right]_{x=\pi/4} + \left[\frac{dv}{dx}\right]_{x=\pi/4}$$

On substituting the values of $\frac{du}{dx}$ and $\frac{dv}{dx}$, we get

$$\left[\frac{dy}{dx}\right]_{x=\pi/4} = 2\log_{1/\sqrt{2}}\left(\frac{1}{\sqrt{2}}\right)\left[\frac{1\times\log\frac{1}{\sqrt{2}} + 1\times\log\frac{1}{\sqrt{2}}}{\left(\log\frac{1}{\sqrt{2}}\right)}\right] + \frac{2}{1+\left(\frac{\pi}{4}\right)^4} = \frac{-8}{\log 2} + \frac{32}{16+\pi^2}$$

Example : 43

If $y = e^{-xz}(x\sqrt{z})$ and $z^4 + x^2z = x^5$, find dy/dx in terms of x and z.

Solution

Consider $z^4 + x^2z - x^5$

Differentiating with respect to x, we get :

$$\left[\frac{dy}{dx}\right]_{x=\pi/4}$$

$$4z^3\frac{dz}{dx} + x^2\frac{dz}{dx} + 2xz = 5x^4$$

$$\Rightarrow \quad \frac{dz}{dx} = \frac{5x^4 - 2xz}{4z^3 + x^2} \quad ....(i)$$

Consider $y = e^{-xz}\sec^{-1}(x\sqrt{z})$

Differentiating with respect to x, we get :

$$\frac{dy}{dx} = e^{-xz}\frac{1}{|x|\sqrt{z}\left(\sqrt{x^2z-1}\right)}\frac{d}{dx}(x\sqrt{z}) + \sec^{-1}x\sqrt{z} \times e^{-xz}\frac{d}{dx}(-xz)$$

$$= e^{-xz}\frac{1}{|x|\sqrt{z}\left(\sqrt{x^2z-1}\right)}\left(\sqrt{z} + x\frac{1}{2\sqrt{z}}\frac{dz}{dx}\right) + e^{-xz}\sec^{-1}x\sqrt{z}\left(-z - x\frac{dz}{dx}\right)$$

On substituting the value of dz/dx from (i), we get

$$= e^{-xz}\left(\frac{1}{|x|\sqrt{x^2z-1}} + \frac{x}{2|x|z\sqrt{x^2z-1}}\frac{x(5x^3-2z)}{4z^3+x^2} - z\sec^{-1}x\sqrt{z} - \sec^{-1}x\sqrt{z}\frac{x^2(5x^3-2z)}{4z^3+x^2}\right)$$

Example : 44

Find f′(x) if f(x) = $\begin{vmatrix} x & x^2 & x^3 \\ 1 & 2x & 3x^2 \\ 0 & 2 & 6x \end{vmatrix}$

Solution

$$f'(x) = \begin{vmatrix} \frac{d}{dx}(x) & \frac{d}{dx}(x^2) & \frac{d}{dx}(x^3) \\ 1 & 2x & 3x^2 \\ 0 & 2 & 6x \end{vmatrix} + \begin{vmatrix} x & x^2 & x^3 \\ \frac{d}{dx}(1) & \frac{d}{dx}(2x) & \frac{d}{dx}(3x^2) \\ 0 & 2 & 6x \end{vmatrix} + \begin{vmatrix} x & x^2 & x^3 \\ 1 & 2x & 3x^2 \\ \frac{d}{dx}(0) & \frac{d}{dx}(2) & \frac{d}{dx}(6x) \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 2x & 3x^2 \\ 1 & 2x & 3x^2 \\ 0 & 2 & 6x \end{vmatrix} + \begin{vmatrix} x & x^2 & x^3 \\ 0 & 2 & 6x \\ 0 & 2 & 6x \end{vmatrix} + \begin{vmatrix} x & x^2 & x^3 \\ 1 & 2x & 3x^2 \\ 0 & 0 & 6 \end{vmatrix}$$

$$= 0 + 0 + \begin{vmatrix} x & x^2 & x^3 \\ 1 & 2x & 3x^2 \\ 0 & 0 & 6 \end{vmatrix} = 6\,(2x^2 - x^2) = 6x^2$$

Example : 45

Differentiate $y = \cos^{-1}\frac{1-x^2}{1+x^2}$ with respect to $z = \tan^{-1}x$. Also discuss the differentiability of this function.

Solution

The given function is $y = \cos^{-1}\frac{1-x^2}{1+x^2}$

Substitute $x = \tan\theta$

$\Rightarrow \quad y = \cos^{-1}\frac{1-\tan^2\theta}{1+\tan^2\theta} = \cos^{-1}(\cos 2\theta)$

$\Rightarrow \quad y = 2\theta = 2\tan^{-1}x \quad$ for $\quad 0 \le 2\theta \le \pi$

$\Rightarrow \quad 0 \le \theta \le \pi/2$

$\Rightarrow \quad 0 \le x < \infty$

and $\quad y = -2\theta = -2\tan^{-1}x \quad$ for $\quad -\pi < 2\theta < 0$

$\Rightarrow \quad -\pi/2 < \theta < 0$

$\Rightarrow \quad -\infty < x < 0$

So the given function reduces to :

$$y = \begin{cases} 2\tan^{-1}x, & x \ge 0 \\ -2\tan^{-1}x, & x < 0 \end{cases}$$

Differentiating with respect to $\tan^{-1}x$, we get

$$\frac{dy}{d(\tan^{-1}x)} = \begin{cases} 2 & x \ge 0 \\ -2 & x < 0 \end{cases}$$

Alternate Method

$$y = \cos^{-1}\frac{1-x^2}{1+x^2}$$

Differentiating with respect to x, we get

$$\frac{dy}{dx} = \frac{-1}{\sqrt{1-\left(\frac{1-x^2}{1+x^2}\right)^2}} \quad \frac{(1+x^2)(-2x)-2x(1-x^2)}{(1+x^2)^2} = \frac{4x}{\sqrt{4x^2}} \quad \frac{1}{1+x^2}$$

In the question, $z = \tan^{-1} x$. On differentiating with respect to x, we get

$$= \frac{1}{1+x^2}$$

On applying chain rule,

$$= \frac{dy/dx}{dz/dx} = \frac{4x}{\sqrt{4x^2}} = \frac{4x}{|2x|} = \begin{cases} 2 & x \geq 0 \\ -2 & x < 0 \end{cases}$$

Differentiability x = 0

LHD = –2 and RHD = 2

As LHD ≠ RHD, f(x) is not differentiable at x = 0

Example : 46

Find dy/dx at x= – 1 when $\sin y^{(\sin(\pi x/2)} + \frac{\sqrt{3}}{2} \sec^{-1}(2x) + 2^x \tan[\ell n(x+2)] = 0$

Solution

The given curve is : $\sin y^{\sin(\pi x/2)} + \frac{\sqrt{3}}{2} \sec^{-1}(2x) + 2^x \tan[\ell n(x+2)] = 0$

Let $A = \sin y^{\sin(px/2)}$; $B = \frac{\sqrt{3}}{2} \sec^{-1}(2x)$ and $C = 2^x \tan[\ell n(x+2)]$

$\Rightarrow$ A + B + C = 0 ....(i)

Consider A

Taking log and then differentiating A w.r.t. x, we get $\frac{dy}{dx}$

$$\frac{1}{A}\,\frac{dA}{dx} = \left[\frac{\pi}{2}\cos\frac{\pi x}{2}\,\ell n(\sin y) + \sin\frac{\pi x}{2}\cot y\frac{dy}{dx}\right]$$

At x = –1

$$\left[\frac{dA}{dx}\right]_{x=-1} = (\sin y)^{-1}\left[0 + (-1)\frac{\cos y}{\sin y}\left(\frac{dy}{dx}\right)_{x=-1}\right] = -\frac{\cos y}{\sin^2 y}\left(\frac{dy}{dx}\right)_{x=-1}$$

Consider B

$$B = \frac{\sqrt{3}}{2} \sec^{-1} 2x$$

Differentiating with respect to x, we get $\frac{dB}{dx} = \frac{\sqrt{3}}{2|x|\sqrt{4x^2-1}}$

At x = – 1 $\quad \left[\frac{dB}{dx}\right]_{x=-1} = \frac{1}{2}$

Consider C

$C = 2^x \tan[\ell n(x+2)]$

Differentiating with respect to x, we get

$$\frac{dC}{dx} = 2^x\,\frac{\sec^2[\ell n(x+2)]}{x+2} + 2^x\,\ell n\,2\tan[\ell n(x+2)]$$

At x = – 1 $\quad \left[\frac{dC}{dx}\right]_{x=-1} = \frac{1}{2}$

Differentiate (i) to get :

$$\frac{dA}{dx} + \frac{dB}{dx} + \frac{dC}{dx} = 0$$

On substituting the values of dA/dx, dB/dx and dC/dx at x = – 1, we get

$$\left[\frac{dy}{dx}\right]_{x=-1} = \frac{\sin^2 y}{\cos y} = \frac{\sin^2 y}{\pm\sqrt{1-\sin^2 y}} \qquad \text{...........(ii)}$$

Finding the value of sin y

Consider the given curve and put x = –1 in it to get

$$(\sin y)^{-1} + \frac{\sqrt{3}}{2}\sec^{-1}(-2) = 0$$

$\Rightarrow$ $\sin y = -\dfrac{2}{\sqrt{3}\sec^{-1}(-2)} = -\dfrac{\sqrt{3}}{\pi}$ [using $\sec^{-1}(-2) = \cos^{-1}(-1/2) = p - \cos^{-1}(1/2) = 2\pi/3$]

Substituting the value of sin y in (2), we get :

$$\left[\frac{dy}{dx}\right]_{x=-1} = \pm\frac{\left(\frac{-\sqrt{3}}{\pi}\right)^2}{\sqrt{1-\left(\frac{-\sqrt{3}}{\pi}\right)^2}} = \pm\frac{3}{\pi\sqrt{\pi^2-3}}$$

Example : 47

If g is the inverse function of f and $f'(x) = \dfrac{1}{1+x^n}$, prove that $g'(x) = 1 + [g(x)]^n$.

Solution

As g is inverse function of f(x), we can take : $g(x) = f^{-1}(x)$

$\Rightarrow$ $f[g(x)] = x$

Differentiating with respect to x, we get : $f'[g(x)]\, g'(x) = 1$

$$\Rightarrow \quad g'(x) = \frac{1}{f'[(g(x)]} = \frac{1}{\frac{1}{1+[g(x)]^n}}$$

$\Rightarrow$ $g'(x) = 1 + [g(x)]^n$

Example : 48

If $y = 1 + \dfrac{z_1}{x-c_1} + \dfrac{c_2x}{(x-c_1)(x-c_2)} + \dfrac{c_3x^2}{(x-c_1)(x-c_2)(x-c_3)}$, then

Show that $\dfrac{dy}{dx} = \dfrac{y}{x}\left[\dfrac{c_1}{c_1-x} + \dfrac{c_2}{c_2-x} + \dfrac{c_3}{c_3-x}\right]$

Solution

$$y = \frac{x}{x-c_1} + \frac{c_2x}{(x-c_1)(x-c_2)} + \frac{c_3x^2}{(x-c_1)(x-c_2)(x-c_3)}$$

$$\Rightarrow \quad y = \frac{x(x-c_2)+c_2x}{(x-c_1)(x-c_2)} + \frac{c_3x^2}{(x-c_1)(x-c_2)(x-c_3)}$$

$$\Rightarrow \quad y = \frac{x^2}{(x-c_1)(x-c_2)} + \frac{c_3x^2}{(x-c_1)(x-c_2)(x-c_3)}$$

$\Rightarrow \quad y = \dfrac{x^3}{(x-c_1)(x-c_2)(x-c_3)}$

Take log on both sides and then differentiate to get

$\log y = 3 \log x - \log (x - c_1) - \log (x - c_2) - \log (x - c_3)$

$$\Rightarrow \quad \frac{dy}{dx} = y\left[\frac{3}{x} - \frac{1}{x-c_1} - \frac{1}{x-c_2} - \frac{1}{x-c_3}\right] = \frac{y}{x}\left[\left(1-\frac{x}{x-c_1}\right)+\left(1-\frac{x}{x-c_2}\right)+\left(1-\frac{x}{x-c_3}\right)\right]$$

$$= \frac{y}{x}\left[\frac{c_1}{c_1-x}+\frac{c_2}{c_2-x}+\frac{c_3}{c_3-x}\right]$$

Example : 49

If $p^2 = a^2 \cos^2\theta + b^2 \sin^2\theta$, the prove that $p + \dfrac{d^2p}{d\theta^2} = \dfrac{a^2b^2}{p^3}$

Solution

$p^2 = a^2 \cos^2\theta + b^2 \sin^2\theta$

$\Rightarrow \quad 2p^2 = a^2 + b^2 + (a^2 - b^2) \cos 2\theta$

$\Rightarrow \quad 2p^2 - a^2 - b^2 = (a^2 - b^2) \cos 2\theta$ ....(ii)

Also $2pp_1 = a^2 (-\sin 2\theta) + b^2 (\sin 2\theta)$ (by taking $p_1 = dp/d\theta$)

$\Rightarrow \quad 2pp_1 = (b^2 - a^2) \sin 2\theta$ ....(ii)

Square (i) and (ii) and add,

$\Rightarrow \quad [2p^2 - (a^2 + b^2)]^2 + 4p^2\, p_1^2\, (a^2 - b^2)^2$

$\Rightarrow \quad 4p^4 + (a^2 + b^2)^2 - (a^2 - b^2) + 4p^2\, p_1^2 = 4p^2 (a^2 + b^2)$

$\Rightarrow \quad p^4 + a^2b^2 + p^2\ \ p_1^2 = p^2 (a^2 + b^2)$

$\Rightarrow \quad p^2 + \dfrac{a^2b^2}{p^2} + p_1^2 = a^2 + b^2$

On differentiating w.r.t. θ, we get

$pp_1 - \dfrac{2a^2b^2}{p^3}\ p_1 + 2p_1\, p_2 = 0$ (by taking $p_2 = d^2p/d\theta^2$)

$\Rightarrow \quad p + p_2 = \dfrac{a^2b^2}{p^3}$

Example : 50

If $y^{1/m} + y^{-1/m} = 2x$, then prove that $(x^2 - 1)\dfrac{d^2y}{dx^2} + x\dfrac{dy}{dx} - m^2y = 0$

Solution

$y^{\frac{1}{m}} + y^{-\frac{1}{m}} = 2x$ ....(i)

Using $\left(y^{\frac{1}{m}} + y^{-\frac{1}{m}}\right)^2 - \left(y^{\frac{1}{m}} + y^{-\frac{1}{m}}\right)^2 = 4$

we get $y^{\frac{1}{m}} + y^{-\frac{1}{m}} = 2\sqrt{x^2-1}$ ....(ii)

Adding (i) and (ii), we get $y^{\frac{1}{m}} = x + \sqrt{x^2-1}$ $\Rightarrow$ $y = \left(x+\sqrt{x^2-1}\right)^m$ ....(iii)

Differentiating wrt, we get

$$y' = m\left(x+\sqrt{x^2-1}\right)^{m-1}\left(1+\frac{x}{\sqrt{x^2-1}}\right)$$

$\Rightarrow$ $$y'\sqrt{x^2-1} = m\left(x+\sqrt{x^2-1}\right)^{m}$$

On squaring above and using (iii), we get $(x^2 - 1)\, y'^2 = m^2y^2$

Differentiate again to get

$$2xy'^2 + (x^2 - 1)\, 2y'y'' = 2m^2\, y\, y'$$

$\Rightarrow$ $$(x^2 - 1)\, y'' + xy' = m^2\, y$$

Hence proved

Example : 51

Evaluate $\lim\limits_{x\to a} \dfrac{x^n - a^x}{x^x - a^a}$ using LH rule $\left(\dfrac{0}{0}\text{type of indeterminate form}\right)$

Solution

Let $L = \lim\limits_{x\to a} \dfrac{x^a - a^x}{x^x - a^a}$ ....(i)

Note that the expression assumes $\dfrac{0}{0}$ type of indeterminate form at x = a.

As the expression satisfies all the conditions of LH rule, we can evaluate this limit by using LH rule.

Apply LH rule on (i) to get :

$$L = \lim_{x\to a} \frac{ax^{n-1} - a^x\,.\log a}{x^x(1+\log x) - 0}$$

$\Rightarrow$ $$L = \frac{a^a - a^a\log a}{a^a(1+\log a)} = \frac{1-\log a}{1+\log a} = \frac{\log\left(\frac{c}{a}\right)}{\log(ae)}$$

Example : 52

Evaluate $\lim\limits_{x\to a}\left(2-\dfrac{a}{x}\right)^{\tan\frac{\pi x}{2a}}$ using LH rule (1^∞ type of indeterminate form)

Solution

Let $y = \lim\limits_{x\to a}\left(2-\dfrac{a}{x}\right)^{\tan\frac{\pi x}{2a}}$ (1^∞ form)

Taking log of both sides, we get :

$$\log y = \lim_{x\to a} \tan\left(\frac{\pi x}{2a}\right)\log\left(2-\frac{a}{x}\right) \qquad (\infty \times 0 \text{ form})$$

$$= \lim_{x\to a} \frac{\log\left(2-\frac{a}{x}\right)}{\cot\left(\frac{\pi x}{2a}\right)} \qquad \left(\frac{0}{0}\text{form}\right)$$

Applying LH rule, we get

$$L = \lim_{x\to a} \frac{\left(\frac{a}{x^2}\right)}{\left(2-\frac{a}{x}\right)\operatorname{cosec}^2\left(\frac{\pi x}{2a}\right)\left(\frac{\pi}{2a}\right)} = \frac{\frac{1}{a}}{(-1)\operatorname{cosec}^2\left(\frac{\pi}{2}\right)\left(\frac{\pi}{2a}\right)} = -\frac{2}{\pi}$$

$\therefore \qquad y = e^{-2/\pi}$

Example : 54

Evaluate $\lim_{x\to 0} \frac{\sin 3x^2}{\ell n\cos(2x^2-1)}$ using LH rule $\qquad \left(\frac{0}{0}\text{type of indeterminate form}\right)$

Solution

Let $L = \lim_{x\to 0} \frac{\sin 3x^2}{\ell n\cos(2x^2-x)}$ (0/0 form)

Apply LH rule to get :

$$L = \lim_{x\to 0} \frac{-6x\cos 3x^2\cos(2x^2-x)}{(4x-1)\sin(2x^2-x)} = -6\lim_{x\to 0}\frac{3x^2\cos(2x^2-x)}{4x-1}\lim_{x\to 0}\frac{x}{(2x^2-x)}$$

The limit of the first factor is computed directly, the limit of the second one, which represents an indeterminate form of the type $\frac{0}{0}$ is found with the aid of the L′Hospital's rule. Again consider,

$$L = -6\lim_{x\to 0}\frac{\cos 3x^2\cos(2x^2-x)}{4x-1}\lim_{x\to 0}\frac{x}{\sin(2x^2-x)}$$

$$\Rightarrow \qquad L = -6 \,.\, \frac{1.1}{-1}\lim_{x\to 0}\frac{1}{(4x-1)\cos(2x^2-x)}$$

$$\Rightarrow \qquad L = -6\,\frac{1}{-1.1} = -6$$

Example : 55

Evaluate $\lim_{x\to\infty}\frac{\log_a x}{x^k}$ (k > 0) using LH rule $\qquad \left(\frac{\infty}{\infty}\text{type of indeterminate form}\right)$

Solution

Let $L = \lim_{x\to\infty}\frac{\log_a x}{x^k}$ (∞/∞ form)

Apply LH rule to get :

$$L = \lim_{x\to+\infty}\frac{\frac{1}{x}\log_a e}{kx^{k-1}}$$

$$\Rightarrow \qquad L = \log_a e\lim_{x\to+\infty}\frac{1}{kx^k} = 0$$

Example : 56

Evaluate $\lim\limits_{x\to 1}\left(\frac{1}{\ell nx}-\frac{1}{x-1}\right)$ using LH rule $\left(\frac{\infty}{\infty}\text{type of indeterminate form}\right)$

Solution

Let $L = \lim\limits_{x\to 1}\left(\frac{1}{\ell nx}-\frac{1}{x-1}\right)$ $(\infty - \infty \text{ form})$

Let us reduce it to an indeterminate form of the type $\frac{0}{0}$

$$L = \lim_{x\to 1}\frac{x-1-\ell nx}{(x-1)\ell nx} \qquad (0/0 \text{ form})$$

Apply LH rule to get :

$$L = \lim_{x\to 1}\frac{1-1/x}{\ell nx+1-1/x}$$

$$\Rightarrow \quad L = \lim_{x\to 1}\frac{x-1}{x\ell nx+x-1}$$

Apply LH rule again

$$\Rightarrow \quad L = \lim_{x\to 1}\frac{1}{\ell n\,x+2} = \frac{1}{2}$$

Example : 57 $(\infty^0$ type of indeterminate form)

Evaluate $\lim\limits_{x\to 0}\left[\ell n\left(+1\sin^2 x\right)\cot \ell n^2(1+x)\right]$ using LH ule.

Solution

Let $L = \lim\limits_{x\to 0}\left[\ell n\left(+1\sin^2 x\right)\cot \ell n^2(1+x)\right]$

We have an indeterminate form of the type 0 . ∞. Let us reduce it to an indeterminate form of the type $\frac{0}{0}$.

$$\Rightarrow \quad L = \lim_{x\to 0}\frac{\ell n(1+\sin^2 x)}{\tan \ell n^2(1+x)} \qquad (0/0 \text{ form})$$

Apply LH rule to get :

$$L = \lim_{x\to 0}\frac{\frac{1}{1+\sin^2 x}\sin 2x}{2\sec^2[\ell n^2(1+x)]\ell n(1+x)\,.\,\frac{1}{1+x}}$$

Simplify to get :

$$L = \lim_{x\to 0}\frac{\sin x}{\ell n(1+x)}$$

Apply LH rule again to get :

$$L = \lim_{x\to 0}\frac{\sin x}{\ell n(1+x)} = \lim_{x\to 0}\frac{\cos x}{\frac{1}{1+x}} = 1$$

Example : 58

Evaluate : $\lim_{x\to 0} (1/x)^{\sin x}$ using LH rule. (∞^0 type of indeterminate form)

Solution

We have an indeterminate form of the type ∞^0

Let $y = (1x)^{\sin x}$;

Taking log on both sides, we get :

$\ell n\, y = \sin x\, \ell n\, (1/x)$

$$\Rightarrow \quad \lim_{x\to +0} \ell n\, y = \lim_{x\to +0} \sin x\, \ell n\,(1/x) \qquad (0, \infty \text{ form})$$

Let us transform it to $\frac{\infty}{\infty}$ to apply LH rule.

$$\lim_{x\to +0} \ell n\, y = \lim_{x\to +0} \frac{-\ell nx}{1/\sin x}$$

Apply LH rule to get :

$$\lim_{x\to +0} \ell n\, y = \lim_{x\to 0} \frac{-1/x}{-(\cos x)/\sin^2 x} = \lim_{x\to 0} \frac{\sin^2 x}{x\cos x} = 0$$

$$\Rightarrow \quad \lim_{x\to +0} y = e^0 = 1$$

Example : 59

Find the values of a, b, c so that $\lim_{x\to 0} \frac{ae^x - b\cos x + ce^{-x}}{x\sin x} = 2$

Solution

$$\text{Let } L = \lim_{x\to 0} \frac{ae^x - b\cos x + ce^{-x}}{x\sin x} \qquad \text{...........(i)}$$

Here as $x \to 0$, denominator approaches 0. So for L to be finite, the numerator must tend to 0.

$a - b + c = 0$ ....(ii)

Apply LH rule on (i) to get :

$$L = \lim_{x\to 0} \frac{ae^x + b\sin x - ce^{-x}}{\sin x + x\cos x}$$

Here as $x \to 0$, the denominator tends to 0 and numerator tends to a – c. For L to be finite,

$a - c = 0$ ....(iii)

Apply LH rule again on L to get :

$$L = \lim_{x\to 0} \frac{ae^x + b\cos x + ce^{-x}}{2\cos x - x\sin x}$$

$$\Rightarrow \quad \frac{a+b+c}{2} = 2 \qquad \Rightarrow \qquad a + b + c = 4 \qquad \text{...........(iv)}$$

Solving equations (ii), (iii) and (iv), we get a = 1, b = 2, c = 1

Example : 60

Evaluate : $\lim\limits_{x\to 0} \dfrac{1-\cos x}{x\log(1+x)}$

Solution

Let L = $\lim\limits_{x\to 0} \dfrac{1-\cos x}{x\log(1+x)}$ (0/0 form)

Using the expansions of cos x and log (1 + x), we get :

$$L = \lim_{x\to 0} \frac{1-\left(1-\frac{x^2}{2!}+\frac{x^4}{4!}-.........\right)}{x\left(x-\frac{x^2}{2}+\frac{x^3}{3}-..........\right)} = \lim_{x\to 0} \frac{\frac{x^2}{2!}-\frac{x^4}{4!}+..........}{x^2-\frac{x^3}{2}+\frac{x^4}{3}-........}$$

Dividing both numerator and denominator by x^2 , we get :

$$L = \lim_{x\to 0} \frac{\frac{1}{2!}-\frac{x^2}{4!}+.......}{1-\frac{x}{2}+\frac{x^2}{3}-........} = \frac{\frac{1}{2}+0-0+........}{1+0-0+.......} = \frac{1}{2}$$

Example : 61

Evaluate : $\lim\limits_{x\to 0} \dfrac{e^x \sin x - x - x^2}{x^2}$

Solution

Let L = $\lim\limits_{x\to 0} \dfrac{e^x \sin x - x - x^2}{x^2}$ (0/0 form)

Using the expansions of sin x and e^x , we get :

$$L = \lim_{x\to 0} \frac{\left(1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}+.....\right)\left(x-\frac{x^3}{3!}+\frac{x^5}{5!}.......\right)-x-x^2}{x^3}$$

$$= \lim_{x\to 0} \frac{x+x^2+\left(\frac{1}{2!}-\frac{1}{3!}\right)x^3+\left(\frac{1}{3!}-\frac{1}{3!}\right)x^4+\left(\frac{1}{4!}-\frac{1}{2!\,.\,3!}+\frac{1}{5!}\right)x^5+.....-x-x^2}{x^3}$$

$$= \lim_{x\to 0} \frac{\frac{1}{3}x^3-\frac{1}{30}x^5+..........}{x^3} = \lim_{x\to 0}\left(\frac{1}{3}-\frac{1}{30}x^2+.......\right) = \frac{1}{2}$$