Example : 1

Find all the tangents to the curve y = cos (x + y), $-2\pi \le x \le 2\pi$ that are parallel to the line x + 2y = 0.

Solution

Slope of tangent (x) = slope of line = – 1/2

$\Rightarrow \quad \frac{dy}{dx} = \frac{-1}{2}$

Differentiating the given equation with respect to x,

$\Rightarrow \quad \frac{dy}{dx} = -\sin(x+y)\left(1+\frac{dy}{dx}\right) = \frac{-\sin(x+y)}{1+\sin(x+y)} = \frac{-1}{2}$

$\Rightarrow \quad 2\sin(x+y) = 1 + \sin(x+y)$

$\Rightarrow \quad \sin(x+y) = 1$

$\Rightarrow \quad x + y = n\pi + (-1)^n \pi/2,\ n \in I$ in the given interval, we have $x + y = \frac{-3\pi}{2}, \frac{\pi}{2}$

(because $-(2\pi + 1) \le x + y \le 2\pi + 1$)

Substituting the value of (x + y) in the given curve i.e. y = cos (x + y), we get

$y = 0$ and $x = \frac{-3\pi}{2}, \frac{\pi}{2}$

Hence the points of contact are $\left(\frac{-3\pi}{2}, 0\right)$ and $\left(\frac{\pi}{2}, 0\right)$ and the slope is $\left(\frac{-1}{2}\right)$

$\Rightarrow$ Equations of tangents are $y - 0 = \frac{-1}{2}\left(x + \frac{3\pi}{2}\right)$ and $y - 0 = \frac{-1}{2}\left(x - \frac{\pi}{2}\right)$

$\Rightarrow \quad 2x + 4y + 3\pi = 0$ and $2x + 4y - \pi = 0$

Example : 2

Find the equation of the tangent to $\frac{x^m}{a^m} + \frac{y^m}{b^m} = 1$ at the point (x_0, y_0)

Solution

Differentiating wrt x,

$\Rightarrow \quad \frac{mx^{m-1}}{a^m} + \frac{my^{m-1}}{b^m}\frac{dy}{dx} = 0$

$\Rightarrow \quad \frac{dy}{dx} = -\frac{b^m}{a^m}\left(\frac{x}{y}\right)^{m-1}$

$\Rightarrow$ at the given point (x_0, y_0), slope of tangent is $\left.\frac{dy}{dx}\right]_{(x_0, y_0)} = -\left(\frac{b}{a}\right)^m \left(\frac{x_0}{y_0}\right)^{m-1}$

$\Rightarrow$ the equation of tangent is $y - y_0 = -\left(\frac{b}{a}\right)^m \left(\frac{x_0}{y_0}\right)^{m-1}(x - x_0)$

$a^m yy_0^{m-1} - a^m y_0^m = -b^m xx_0^{m-1} + b^m x_0^m$

$a^m yy_0^{m-1} + b^m x x_0^{m-1} = a^m y_0^m + b^m x_0^m$

using the equation of given curve, the right side can be replaced by $a^m b^m$

$\therefore \quad a^m yy_0^{m-1} + b^m x x_0^{m-1} = a^m b^m$

$\Rightarrow$ the equation of tangent is

$\frac{x}{a}\left(\frac{x_0}{a}\right)^{m-1} = \frac{y}{b}\left(\frac{y_0}{b}\right)^{m-1} = 1$

Note : The result of this example can be very useful and you must try remember it

Example : 3

Find the equation of tangent to the curve $x^{2/3} + y^{2/3} = a^{2/3}$ at $(x_0 - y_0)$. Hence prove that the length of the portion of tangent intercepted between the axes is constant.

Solution

Method 1 :

$$\frac{2}{3}\, x^{\frac{-1}{3}} + \frac{2}{3}\, y^{\frac{-1}{3}}\, \frac{dy}{dx} = 0$$

$$\Rightarrow \quad \left.\frac{dy}{dx}\right]_{(x_0,\, y_0)} = -\left(\frac{y_0}{x_0}\right)^{\frac{1}{3}}$$

$$\Rightarrow \quad \text{equation is } y - y_0 = -\left(\frac{y_0}{x_0}\right)^{\frac{1}{3}} (x - x_0)$$

$\Rightarrow \quad x_0^{1/3}\, y - y_0\, x_0^{1/3} = -\, xy_0^{1/3} + x_0\, y_0^{1/3}$

$\Rightarrow \quad x\, y_0^{1/3} + yx_0^{1/3} = x_0\, y_0^{1/3} + y_0\, x_0^{1/3}$

$$\Rightarrow \quad \frac{xy_0^{1/3}}{x_0^{1/3}y_0^{1/3}} + \frac{yx_0^{1/3}}{x_0^{1/3}y_0^{1/3}} = x_0^{2/3} + y_0^{2/3}$$

$$\Rightarrow \quad \text{equation of tangent is : } \frac{x}{x_0^{1/3}} + \frac{y}{y_0^{1/3}} = a^{2/3}$$

Length intercepted between the axes :

$$\text{length} = \sqrt{(x\ \text{intercept})^2 + (y\ \text{intercept})^2}$$

$$= \sqrt{\left(x_0^{1/3}a^{2/3}\right)^2 + \left(y_0^{1/3}a^{2/3}\right)^2}$$

$$= \sqrt{x_0^{2/3}a^{4/3} + y_0^{2/3}a^{4/3}}$$

$$= a^{2/3}\sqrt{x_0^{2/3} + y_0^{2/3}}$$

= a i.e. constant

Method 2

Express the equation in parametric form

$x = a \sin^2 t, \qquad y = a \cos^3 t$

Equation of tangent is :

$$y - a\cos^3 t = \frac{-3a\cos^2 t \sin t}{3a\sin^2 t \cos t}\,(x - a\sin^3 t)$$

$\Rightarrow \quad y \sin - a \sin t \cos^3 t = -\, x \cos t - a \sin^3 t \cos t$

$\Rightarrow \quad x \cos t + y \sin t = a \sin t \cos t$

$$\Rightarrow \quad \frac{x}{\sin t} + \frac{y}{\cos t} = a$$

in terms of (x_0, y_0) equation is :

$$\frac{x}{(x_0/a)^{1/3}} + \frac{y}{(y_0/a)^{1/3}} = a$$

Length of tangent intercepted between axes = $\sqrt{(x_{int})^2 + (y_{int})^2} = \sqrt{a^2\sin^2 t + a^2\cos^2 t} = a$

Note :

1. The parametric form is very useful in these type of problems

2. Equation of tangent can also be obtained by substituting b = a and m = 2/3 in the result of example 2

Example : 4

For the curve $xy = c^2$, prove that

(i) the intercept between the axes on the tangent at any point is bisected at the point of contact.

(ii) the tangent at any point makes with the co-ordinate axes a triangle of constant area.

Solution

Let the equation of the curve in parametric form by x = ct, y = c/t

Let the point of contact be (ct, c/t)

Equation of tangent is :

$$y - c/t = \frac{-c/t^2}{c}(x - ct)$$

$\Rightarrow$ $t^2y - ct = -x + ct$

$\Rightarrow$ $x + t^2y = 2\,ct$ ....(i)

(i) Let the tangent cut the x and y axes at A and B respectively

Writing the equations as : $\frac{x}{2ct} + \frac{y}{2ct/t} = 1$

$\Rightarrow$ $x_{intercept} = 2\,ct$, $y_{intercept} = 2c/t$

$\Rightarrow$ $A \equiv (2ct = 0$, and $B \equiv \left(0, \frac{2c}{1}\right)$

mid point of AB $\equiv \left(\frac{2ct+0}{2}, \frac{0+2c/t}{2}\right) \equiv (ct, c/t)$

Hence, the point of contact bisects AB

(ii) If O is the origin, Area of triangle $\Delta OAB = 1/2\,(OA)\,(OB) = \frac{1}{2}(2ct)\frac{(2c)}{1} = 2c^2$

i.e. constant for all tangents because it is independent of t.

Example : 5

Find the abscissa of the point on the curve $ay^2 = x^3$, the normal at which cuts of equal intercept from the axes.

Solution

The given curve is $ay^2 = x^3$ ....(i)

Differentiate to get :

$$2ay\frac{dy}{dx} = 3x^2$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{3x^2}{2ay}$$

The slope of normal = $\frac{1}{-\frac{dy}{dx}} = -\frac{2ay}{3x^2}$

since the normal makes equal intercepts on the axes, its inclination to axis of x is either 45º or 135º.

So two normal are possible with slopes 1 and – 1

$$\Rightarrow \quad -\frac{2ay}{3x^2} = \pm 1$$

On squaring $4a^2y^2 = 9x^4$

Using (i), we get : $4a\,x^3 = 9x^4$

$\Rightarrow$ $x = 4a/9$

Example : 6

Show that two tangents can be drawn from the point A(2a, 3a) to the parabola $y^2 = 4ax$. Find the equations of these tangents.

Solution

The parametric form for $y^2 = 4ax$ is $x = at^2$, $y = 2at$

Let the point $P(at^2, 2at)$ on the parabola be the point of contact for the tangents drawn form A

i.e. $y - 2at = \frac{2a}{2at}(x - at^2)$

$\Rightarrow$ $ty - 2at^2 = x - at^2$

$\Rightarrow$ $x - ty + at^2 = 0$ ....(i)

it passes through A(2a, 3a)

$\Rightarrow$ $2a - 3at + at^2 = 0$

$\Rightarrow$ $t^2 - 3t + 2 = 0$

$\Rightarrow$ $t = 1, 2$

Hence there are two points of contact P_1 and P_2 corresponding to $t_1 = 1$ and $t_2 = 2$ on the parabola. This means that two tangents can be drawn.

Using (i), the equations of tangents are :

$x - y + a = 0$ and $x - 2y + 4a = 0$

Example : 7

Find the equation of the tangents drawn to the curve $y^2 - 2x^3 - 4y + 8 = 0$ from the point (1, 2)

Solution

Let tangent drawn from (1, 2) to the curve

$y^2 - 2x^3 - 4y + 8 = 0$ meets the curve in point (h, k)

Equation of tangents at (h, k)

Slope of tangent at (h, k)

$$= \left.\frac{dy}{dx}\right]_{(h,k)} = \left.\frac{3x^2}{y-2}\right]_{(h,k)} = \frac{3h^2}{k-2}$$

Equation of tangent is $y - k = \frac{3h^2}{k-2}(x - h)$

As tangent passes through (1, 2), we can obtain $2 - k = \frac{3h^2}{k-2}(1 - h)$

$\Rightarrow$ $3h^3 - 3h^2 - k^2 + 4k - 4 = 0$ ....(i)

As (h, k) lies on the given curve, we can make

$k^2 - 2h^3 - 4k + 8 = 0$ ....(ii)

Adding (i) and (ii), we get $h^3 - 3h^2 + 4 = 0$

$\Rightarrow$ $(h + 1)(h - 2)^2 = 0$

$\Rightarrow$ $h = -1$ and $h = 2$

For $h = -1$, k is imaginary

So consider only $h = 2$.

Using (ii) and h = 2, we get $k = 2 \pm 2\sqrt{3}$.

$(2, 2 + 2\sqrt{3}) =$ and $(2, 2) - 2\sqrt{3})$

Equation of tangents at these points are :

$y - (2 + 2\sqrt{3}) = 2\sqrt{3}(x - 2)$

and $y - (2 - \sqrt{3}) = -2\sqrt{3}(x - 2)$

Example : 8

Find the equation of the tangent to $x^3 = ay^3$ at the point A (at^2, at^3). Find also the point where this tangent meets the curve again.

Solution

Equation of tangent to : $x = at^2$, $y = at^3$ is

$$y - at^3 = \frac{3at^3}{2at}(x - at^2)$$

$\Rightarrow \quad 2y - 2at^3 = 3tx - 3at^3$

i.e. $\quad 3tx - 2y - at^3 = 0$

Let B (at_1^2 , at_1^3) be the point where it again meets the curve.

$\Rightarrow \quad$ slope of tangent at A = slope of AB $\frac{3at^2}{2at} = \frac{a(t^3 - t_1^3)}{a(t^3 - t_1^2)}$

$$\Rightarrow \quad \frac{3t}{2} = \frac{t^2 + t_1^2 + tt_1}{t + t_1}$$

$\Rightarrow \quad 3t^2 + 3tt_1 = 2t^2 + 2t_1^2 + 2t\,t_1$

$\Rightarrow \quad 2t_1^2 - t\,t_1 - t^2 = 0$

$\Rightarrow \quad (t_1 - t)(2t_1 + t) = 0$

$\Rightarrow \quad t_1 = t \quad$ or $\quad t_1 = -1/2$

The relevant value is $t_1 = -t/2$

Hence the meeting point B is = $\left[a\left(\frac{-t}{2}\right)^2, a\left(\frac{-t}{2}\right)^3\right] = \left[\frac{at^2}{4}, \frac{-at^3}{8}\right]$

Example : 9

Find the condition that the line $x \cos\alpha + y \sin\alpha = P$ may touch the curve $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

Solution

Let (x_1, y_1) be the point of contact

$\Rightarrow \quad$ the equation of tangent is $y - y_1 \left(\frac{dy}{dx}\right)_{(x_1, y_1)} (x - x_1)$

$$\Rightarrow \quad y - y_1 = \frac{-b^2x_1}{a^2y_1}(x - x_1)$$

$\Rightarrow \quad a^2 yy_1 - a^2 y_1^2 = -b^2 x x_1 + b^2 x_1^2$

$\Rightarrow \quad b^2 x x_1 + a^2 y y_1 = b^2 x_1^2 + a^2 y_1^2$

Using the equation of the curve : $\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1$ is the tangent

If this tangent and the given line coincide, then the ratio of the coefficients of x and y and the constant terms must be same

Comparing $x \cos\alpha + y \sin\alpha = P$ and $\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1$

we get $\frac{\cos\alpha}{x_1/a^2} = \frac{\sin\alpha}{y_1/b^2} = \frac{P}{1}$

$\Rightarrow \quad Px_1 = a^2 \cos\alpha$, $Py_1 = b^2 \sin\alpha$ and also we have $\frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} = 1$

From these three equations, we eliminate x_1, y_1 to get the required condition.

$$\frac{1}{a^2}\left(\frac{a^2\cos\alpha}{P}\right)^2 + \frac{1}{b^2}\left(\frac{b^2\sin\alpha}{P}\right)^2 = 1$$

$\Rightarrow \quad a^2\cos^2\alpha + b^2\sin^2\alpha = P^2$

Example : 10

Find the condition that the curves ; $ax^2 + by^2 = 1$ $a'x^2 + b'y^2 = 1$ may cut each other orthogonally (at right angles)

Solution

Condition for orthogonality implies that the tangents to the curves at the point of intersection are perpendicular. If (x_0, y_0) is the point of intersection, and m_1, m_2 are slopes of the tangents to the two curves at this point, the $m_1 m_2 = -1$.

Let us find the point of intersection. Solving the equations simultaneously,

$ax^2 + by^2 - 1 = 0$

$a' x^2 + b' y^2 - 1 = 0$

$$\Rightarrow \quad \frac{x^2}{-b+b'} = \frac{y^2}{-a+a'} = \frac{1}{ab'-a'b}$$

$\Rightarrow$ the point of intersection $(x_0 - y_0)$ is given by

$$x_0^2 = \frac{b'-b}{ab'-a'b} \text{ and } y_0^2 = \frac{a-a'}{ab'-a'b}$$

The slope of tangent to the curve $ax^2 + by^2 = 1$ is

$m_1 = \frac{dy}{dx} = \frac{-ax_0}{by_0}$ and the slope of tangent to the curve $a' x^2 + b' y^2 = 1$ is $m_2 = \frac{-a'x_0}{b'y_0}$

for orthogonality, $m_1m_2 = \frac{aa'}{bb'}\frac{x_0^2}{y_0^2} = -1$

Using the values of x_0 and y_0, we get

$$\Rightarrow \quad \frac{aa'}{b'b}\frac{b'-b}{a-a'} = -1$$

$$\Rightarrow \quad \frac{b'-b}{bb'} = \frac{a'-a}{aa'}$$

$$\Rightarrow \quad \frac{1}{b} - \frac{1}{b'} = \frac{1}{a} - \frac{1}{a'} \text{ is the required condition}$$

Example : 11

The equation of two curves are $y^2 = 2x$ and $x^2 = 16y$

(a) Find the angle of intersection of two curves

(b) Find the equation of common tangents to these curves.

Solution

(a) First of all solve the equation of two curves to get their points of intersection.

The two curves are $y^2 = 2x$ ....(i)

and $x^2 = 16y$ ....(ii)

On solving (i) and (ii) two points of intersection are (0, 0) and (8, 4)

At (0, 0)

The two tangents to curve $y^2 = 2x$ and $x^2 = 16y$ are $x = 0$ and $y = 0$ respectively.

So angle between curve = angle between tangents = $\pi/2$

At (8, 4)

Slope of tangent to $y^2 = 2x$ is $m_1 = \left.\frac{dy}{dx}\right]_{at\ (8,\ 4)} = \frac{1}{y}$

$\Rightarrow \quad m_1 = 1/4$

Similarly slope of tangent to $x^2 = 16y$ is $m_2 = 1$

Acute angle between the two curve at (8, 4)

$$= \left| \tan^{-1}\left[\frac{m_1 - m_2}{1 + m_1 m_2}\right] \right| = \left| \tan^{-1} \frac{\frac{1}{4} - 2}{1 + \frac{2}{4}} \right| = \tan^{-1} \frac{4}{5}$$

(b) Let common tangent meets $y^2 = 2x$ in point P whose coordinates are $(2t^2, 2t)$

Equation of tangent at P is $y - 2t = \frac{1}{2t}(x - 2t^2)$

$\Rightarrow \quad 2ty - x = 2t^2$

On solving equation of second curve and tangent (i), we get :

$2t(x^2/16) - x = 2t^2$

$\Rightarrow \quad tx^2 - 8x = 16t^2$

This quadratic equation in x should have equal roots because tangent (i) is also tangent to second curve and hence only one point of intersection.

$\Rightarrow \quad D = 0 \qquad \Rightarrow \quad 64 + 64t^3 = 0$

$\Rightarrow \quad t = -1$

So equation of common tangent can be obtained by substituting $t = -1$ in (i) i.e.

$-2y - x = 2 \qquad \Rightarrow \quad 2y + x + 2 = 0$

Example : 12

Find the intervals where $y = \frac{3}{2}x^4 - 3x^2 + 1$ is increasing or decreasing

Solution

$dy/dx = 6x^3 - 6x = 6x(x - 1)(x + 1)$

This sign of dy/dx is positive in the interval :

$(-1, 0) \cup (1, \infty)$ and negative in the interval : $(-\infty, -1) \cup (0, 1)$

Hence the function is increasing in $[-1, 0] \cup [1, \infty)$ and decreasing $(-\infty, -1] \cup [0, 1]$

Example : 13

Find the intervals where y = cos x is increasing or decreasing

Solution

$$\frac{dy}{dx} = -\sin x$$

Hence function is increasing in the intervals where sin x is negative and decreasing where sin x is positive

$$\frac{dy}{dx} < 0 \quad \text{if} \quad 2n\pi < x < (2n + 1)\pi$$

and $\frac{dy}{dx} > 0$ if $(2n + 1)\pi < x < (2n + 2)\pi$

where n is an integer

Hence the function is increasing in $[(2n + 1)\pi, (2n + 2)\pi]$

and decreasing in $[2n\pi, (2n + 1)\pi]$

Example : 14

Show that $\sin x < x < \tan x$ for $0 < x < \pi/2$.

Solution

We have to prove two inequalities; $x > \sin x$ and $\tan x > x$.

Let $f(x) = x - \sin x$

$f'(x) = 1 - \cos x = 2\sin^2 x/2$

$\Rightarrow \quad f'(x)$ is positive

$\Rightarrow \quad f(x)$ is increasing

By definition, x > 0

$\Rightarrow$ f(x) > f(0)

$\Rightarrow$ x – sin x > 0 – sin 0

$\Rightarrow$ x – sin x > 0

$\Rightarrow$ x > sin x ....(i)

Now, let g(x) = tan x – x

$g'(x) = \sec^2 x - 1 = \tan^2 x$ which is positive

$\Rightarrow$ g(x) is increasing

By definition, x > 0 $\Rightarrow$ g(x) > g(0)

$\Rightarrow$ tan x – x > tan 0 – 0

$\Rightarrow$ tan x – x > 0

$\Rightarrow$ tan x > x ....(ii)

Combining (i) and (ii), we get sin x < x < tan x

Example : 15

Show that x / (1 + x) < log (1 + x) < x for x > 0.

Solution

Let $f(x) = \log(1 + x) - \frac{x}{1+x}$

$$f'(x) = \frac{1}{1+x} - \frac{(1+x)-x}{(1+x)^2}$$

$$f'(x) = \frac{x}{(1+x)^2} \;.\; 0 \text{ for } x > 0$$

$\Rightarrow$ f(x) is increasing

Hence x > 0 $\Rightarrow$ f(x) > f(0) by the definition of the increasing function.

$$\Rightarrow \quad \log(1 + x) - \frac{x}{1+x} > \log(1 + 0) - \frac{0}{1+0}$$

$$\Rightarrow \quad \log(1 + x) - \frac{x}{1+x} > 0$$

$$\Rightarrow \quad \log(1 + x) > \frac{x}{1+x} \quad \text{............(i)}$$

Now let g(x) = x – log (1 + x)

$$g'(x) = 1 - \frac{1}{1+x} = \frac{x}{1+x} > 0 \text{ for } x > 0$$

$\Rightarrow$ g(x) is increasing

Hence x > 0 $\Rightarrow$ g(x) > g(0)

$\Rightarrow$ x – log (1 + x) > 0 – log (1 + 0)

$\Rightarrow$ x – log (1 + x) > 0

$\Rightarrow$ x > log (1 + x) ....(ii)

Combining (i) and (ii), we get

$$\frac{x}{1+x} < \log(1 + x) < x$$

Example : 16

Show that : $x - \frac{x^3}{6} < \sin x$ for $0 < x < \frac{\pi}{2}$

Solution

Let $f(x) = \sin x - x + \frac{x^3}{6}$

$f'(x) = \cos x - 1 + \frac{x^2}{2}$

$f''(x) = -\sin x + x$

$f'''(x) = -\cos x + 1 = 2\sin^2\frac{x}{2} > 0$

$\Rightarrow$ $f''(x)$ is increasing

Hence $x > 0 \Rightarrow f''(x) > f''(0)$

$\Rightarrow$ $-\sin x + x > -\sin 0 + 0$

$\Rightarrow$ $-\sin x + x > 0$

$\Rightarrow$ $f''(x) > 0$

$\Rightarrow$ $f'(x)$ is increasing

Hence $x > 0 \Rightarrow f'(x) > f'(0)$

$\Rightarrow$ $\cos x - 1 + x^2/2 > \cos 0 - 1 + 0/2$

$\Rightarrow$ $\cos x - 1 + x^2/2 > 2$

$\Rightarrow$ $f'(x) > 0$

$\Rightarrow$ $f(x)$ is increasing

Hence $x > 0 \Rightarrow f(x) > f(0)$

$\Rightarrow$ $\sin x - x + x^3/6 > \sin 0 - 0 + 0/6$

$\Rightarrow$ $\sin x - x + x^3/6 > 0$

$\Rightarrow$ $\sin x > x - x^3/6$

Example ; 17

Show that $x \geq \log(1 + x)$ for all $x \in (-1, \infty)$

Solution

Let $f(x) = x - \log(1 + x)$

Differentiate f(x) w.r.t. x to get,

$f'(x) = 1 - \frac{1}{1+x} = \frac{x}{1+x}$

Note that x = 0 is a critical point of $f'(x)$ in $(-1, \infty)$.

So divide the interval about x = 0 and make two cases

Case – I $x \in (-1, 0)$

In this interval, $f'(x) < 0$

$\Rightarrow$ f(x) is a decreasing function

Therefore, $-1 < x < 0 \Rightarrow f(x) \geq f(0) = 0$

Hence $x - \log(1 + x) \geq 0$ for all $x \in (-1, 0)$ ....(i)

Case – II $x \in [0, \infty)$

In this interval, $f'(x) > 0$

$\Rightarrow$ f(x) is an increasing function.

Therefore, $0 \leq x < \infty \Rightarrow f(x) \geq f(0) = 0$

Hence $x - \log(1 + x) \geq 0$ for all $x \in [0, \infty)$ ....(ii)

Combining (i) and (ii), $x \geq \log(1 + x)$ for all $x \in (-1, \infty)$

Example : 18

Find the intervals of monotonicity of the function f(x) = $\frac{|x-1|}{x^2}$

Solution

The given function f(x) can be written as :

$$f(x) = \frac{|x-1|}{x^2} = \begin{cases} \frac{1-x}{x^2} & ; \quad x < 1, x \neq 0 \\ \frac{x-1}{x^2} & ; \quad x \geq 1 \end{cases}$$

Consider x < 1 $f'(x) = \frac{-2}{x^3} + \frac{1}{x^2} = \frac{x-2}{x^3}$

For increasing, $f'(x) > 0 \Rightarrow \frac{x-2}{x^3} > 0$

$\Rightarrow$ $x(x-2) > 0$ ($\because x^4$ is always positive)

$\Rightarrow$ $x \in (-\infty, 0) \cup (2, \infty)$

Combining with x < 1, we get f(x) is increasing in x < 0 and decreasing in $x \in (0, 1)$ ....(i)

Consider $x \geq 1$

$$f'(x) = \frac{-1}{x^2} + \frac{2}{x^3} = \frac{2-x}{x^3}$$

For increasing $f'(x) > 0$

$\Rightarrow$ $(2-x) > 0$ ($\because x^3$ is positive)

$\Rightarrow$ $(x-2) < 0$

$\Rightarrow$ $x < 2$

combining with x > 1, f(x) is increasing in $x \in (1, 2)$ and decreasing in $x \in (2, \infty)$ ....(ii)

Combining (i) and (ii), we get

f(x) is strictly increasing on $x \in (-\infty, 0) \cup (1, 2)$ and strictly decreasing on $x \in (0, 1) \cup (2, \infty)$

Example : 19

Prove that $(a + b)^n \leq a^n + b^n$, a > 0, b > 0 and $0 \leq n \leq 1$

Solution

We want to prove that $(a + b)^n \leq a^n + b^n$ i.e. $\left(\frac{a}{b}+1\right)^n \leq \left(\frac{a}{b}\right)^n + 1$

i.e. $(x + 1)^n \leq 1 + x^n$ where x = a/b and x > 0,

since a and b both are positive.

To prove above inequality, consider

$f(x) = (x + 1)^n - x^n - 1$

Differentiate to get,

$$f'(x) = n(x = 1)^{n-1} - nx^{n-1} = \left[\frac{1}{(x+1)^{1-n}} - \frac{1}{x^{1-n}}\right] \quad ....(i)$$

consider x + 1 > x

$\Rightarrow$ $(x + 1)^{1-n} > x^{1-n}$ ($\because 1 - n > 0$)

$$\Rightarrow \quad \frac{1}{(x+1)^{1-n}} , \frac{1}{x^{1-n}}$$

Combining (i) and (ii), we can say $f'(x) < 0$

$\Rightarrow$ f(x) is a decreasing function $\forall$ x > 0

Consider $x \geq 0$

$f(x) \leq f(0)$ $\because$ f(x) is a decreasing function

$\Rightarrow$ $f(x) \leq 0$

$\Rightarrow$ $(x + 1)^n - x^n - 1 \leq 0$

$\Rightarrow$ $(x + 1)^n \leq x^n + 1$ Hence proved

Example : 20

Find the local maximum and local minimum values of the function $y = x^x$.

Solution

Let $f(x) = y = x^x$

$\Rightarrow \quad \log y = x \log x$

$$\Rightarrow \quad \frac{1}{y}\frac{dy}{dx} = x\,\frac{1}{x} + \log x$$

$$\Rightarrow \quad \frac{dy}{dx} = x^x\,(1 + \log x)$$

$f'(x) = 0 \quad \Rightarrow \quad x^x\,(1 + \log x) = 0$

$\Rightarrow \quad \log x = -1 \quad \Rightarrow \quad x = e^{-1} = 1/e$

Method – I

$f'(x) = x^x\,(1 + \log x)$

$f'(x) = x^x \log ex$

$x < 1/e \;\; ex < 1 \quad \Rightarrow \quad f'(x) < 0$

$x > 1/e \;\; ex > 1 \quad \Rightarrow \quad f'(x) > 0$

The sign of $f'(x)$ changes from –ve to +ve around $x = 1/e$. In other words $f(x)$ changes from decreasing to increasing at $x = 1/e$

Hence $x = 1/e$ is a point of local maximum

Local minimum value $= (1/e)^{1/e} = e^{-1/e}$.

Method – II

$$f''(x) = (1 + \log x)\,\frac{d}{dx}\,x^x + x^x\left(\frac{1}{x}\right) = x^x\,(1 + \log x)^2 + x^{x-1}$$

$f''\,(1/e) = 0 + (e)^{(e-1)/e} > 0.$

Hence $x = 1/e$ is a point of local minimum

Local minimum value is $(1/e)^{1/e} = e^{-1/e}$.

Note ; We will apply the second derivative test in most of the problems.

Example : 21

Let $f(x) = \sin^3 x + \lambda \sin^2 x$ where $-\pi/2 < x < \pi/2$. Find the interval in which λ should lie in order that $f(x)$ has exactly one minimum and exactly one maximum.

Solution

$f(x) = \sin^3 x + \lambda \sin^2 x.$

$f'(x) = 3 \sin^2 x \cos x + 2 \sin x \cos x \times \lambda$

$$f'(x) = 0 \quad \Rightarrow \quad 3 \sin x \cos x \left(\sin x + \frac{2\lambda}{3}\right) = 0$$

$$\Rightarrow \quad \sin x = 0 \quad \text{or} \quad \cos x = 0 \quad \text{or} \quad \sin x = \frac{-2\lambda}{3}$$

$\cos x = 0$ is not possible in the given interval.

$\Rightarrow \quad x = 0$ and $x = \sin^{-1}(-2\lambda/3)$ are two possible values of x.

These represent two distinct values of x if :

(i) $\quad \lambda \neq 0$ because otherwise $x = 0$ will be the only value

(ii) $\quad -1 < -2\lambda/3 < 1 \quad \Rightarrow \quad 3/2 > \lambda > -3/2$

for exactly one maximum and only one minimum these conditions must be satisfied by λ

$$\text{i.e.} \quad \lambda \in \left(-\frac{3}{2}, 0\right) \cup \left(0, \frac{3}{2}\right)$$

Since $f(x)$ is continuous and differentiable function, these can not be two consecutive points of local maximum or local minimum. These should be alternate.

Hence $f'(x) = 0$ at two distinct points will mean that one is local maximum and the other is local minimum.

Example : 23

A windows is in the form of a rectangle surmounted by a semi-circle. The total area of window is fixed. What should be the ratio of the areas of the semi-circular part and the rectangular part so that the total perimeter is minimum?

Solution

Let A be the total area of the window. If 2x be the width of the rectangle and y be the height.

Let 2x be the width of the rectangle and y be the height. Let the radius of circle be x.

$\Rightarrow \quad A = 2xy + \pi/2\ x^2$

Perimeter (P) = $2x + 2y + \pi x$

A is fixed and P is to be minimised

Eliminating y,

$$P(x) = 2x + \pi x + \frac{1}{x}\left(A - \frac{\pi x^2}{2}\right)$$

$P'(x) = 2 + \pi - A/x^2 - \pi/2$

$P'(x) = 0 \quad \Rightarrow \quad x = \sqrt{\frac{2A}{\pi + 4}}$

$P''(x) = 2A/x^3 > 0$

$\Rightarrow$ Perimeter is minimum for $x = \sqrt{\frac{2A}{\pi + 4}}$

for minimum perimeter,

area of semicircle = $\frac{\pi(2A)}{(\pi + 4)2} = \frac{\pi A}{\pi + 4}$

area of rectangle = $A - \frac{\pi A}{\pi + 4} = \frac{4A}{\pi + 4}$

$\Rightarrow$ ratio of the areas of two parts = $\frac{\pi}{4}$

Example : 23

A box of constant volume C is to be twice as long as it is wide. The cost per unit area of the material on the top and four sides faces is three times the cost for bottom. What are the most economical dimensions of the box?

Solution

Let 2x be the length, x be the width and y be the height of the box.

Volume = C = $2x^2y$.

Let then cost of bottom = Rs. k per sqm.

Total cost = cost of bottom + cost of other faces

$= k(2x^2) + 3x\ (4xy + 2xy + 2x^2) = 2k$

$= 2k\ (4x^2 + 9xy)$

Eliminating y using C = $2x^2y$,

Total cost = $2k\ (4x^2 + 9C/2x)$

Total cost is to be minimised.

Let total cost = $f(x) = 2k\left(4x^2 + \frac{9C}{2x}\right)$

$$f'(x) = 2k\left(8x - \frac{9c}{2x^2}\right)$$

$f'(x) = 0 \quad \Rightarrow \quad 8x - \frac{9c}{2x^2} = 0$

$\Rightarrow \quad x = \left(\frac{9C}{16}\right)^{1/3}$

$$f''(x) = 2k\left(8 + \frac{9C}{x^3}\right) > 0$$

hence the cost is minimum for $x = \left(\frac{9C}{16}\right)^{1/3}$ and $y = \frac{C}{2x^2} = \frac{C}{2}\left(\frac{16}{9C}\right)^{2/3} = \left(\frac{32C}{81}\right)^{1/3}$

The dimensions are : $2\left(\frac{9C}{16}\right)^{1/3}, \left(\frac{9C}{16}\right)^{1/3}, \left(\frac{32C}{81}\right)^{1/3}$

Example : 34

Show that the semi-vertical angle of a cone of given total surface and maximum volume is $\sin^{-1} 1/3$.

Solution

Let r and h be the radius and height of the cone and ℓ be the slant height of the cone.

Total surface area = $S = \pi r\ell + \pi r^2$ ....(i)

Volume = $V = \pi/3 r^2 h$ is to be maximised

Using, $\ell^2 = r^2 + h^2$ and $S = \pi r\ell + \pi r^2$

$$V = \frac{\pi}{3} r^2 \sqrt{\ell^3 - r^2}$$

$$\Rightarrow \quad V = \frac{\pi}{3} r^2 \sqrt{\left(\frac{S - \pi r^2}{\pi r}\right)^2 - r^2}$$

$$\Rightarrow \quad V = \frac{\pi}{3} r^2 \sqrt{\frac{S^2}{\pi^2 r^2} - \frac{2S}{\pi}}$$

We will maximise V^2

Let $V^2 = f(r) = \frac{\pi^2}{9} r^4 \left(\frac{S^2}{\pi^2 r^2} - \frac{2S}{\pi}\right) = f(r) = \frac{S}{9}(Sr^2 - 2\pi r^4)$

$\Rightarrow \quad f'(r) = 0 \Rightarrow \quad 2Sr - \pi r^3 = 0$

$\Rightarrow \quad r = \sqrt{\frac{S}{4\pi}}$ ....(ii)

$$f''(r) = \frac{S}{9}(2S - 24\pi r^2)$$

$$f''\left(\sqrt{\frac{S}{4\pi}}\right) = \frac{S}{9}(2S - 6S) < 0$$

Hence the volume is maximum for $r = \sqrt{\frac{S}{4\pi}}$

To find the semi-vertical angle, eliminate S between (i) and (ii), to get :

$4\pi r^2 = \pi r\ell + \pi r^2$

$\Rightarrow \quad \ell = 3r$

$\sin\theta = r/\ell = 1/3$

$\Rightarrow \quad \theta = \sin^{-1}(1/3)$ for maximum volume.

Example : 25

Find the maximum surface area of a cylinder that can be inscribed in a given sphere of radius R.

Solution

Let r be the radius and h be the height of cylinder. Consider the right triangle shown in the figure.

$2r = 2R \cos\theta$ and $h = 2R \sin\theta$

Surface area of the cylinder = $2\pi rh + 2pr^2$

$\Rightarrow \quad S(\theta) = 4\pi R^2 \sin\theta \cos\theta + 2\pi R^2 \cos^2\theta$

$\Rightarrow \quad S(\theta) = 2\pi R^2 \sin 2\theta + 2\pi R^2 \cos^2\theta$

$\Rightarrow \quad S'(\theta) = 4\pi R^2 \cos 2\theta - 2\pi R^2 \sin 2\theta$

$S'(\theta) = 0 \quad \Rightarrow \quad 2\cos 2\theta - \sin 2\theta = 0$

$\Rightarrow \quad \tan 2\theta = 2 \quad \Rightarrow \quad \theta = \theta_0 = 1/2 \tan^{-1} 2$

$S''(\theta) = -8\pi R^2 \sin 2\theta - 4\pi R^2 \cos 2\theta$

$$S''(\theta_0) = -8\pi R^2 \left(\frac{2}{\sqrt{5}}\right) - 4\pi R^2 \left(\frac{1}{\sqrt{5}}\right) < 0$$

Hence surface area is maximum for $\theta = \theta_0 = 1/2 \tan^{-1} 2$

$$\Rightarrow \quad S_{max} = 2\pi R^2 \left(\frac{2}{\sqrt{5}}\right) + 2\pi R^2 \left(\frac{1+1/\sqrt{5}}{2}\right)$$

$$\Rightarrow \quad S_{max} = \pi R^2 \left(1+\sqrt{5}\right)$$

Example : 26

Find the semi-vertical angle of the cone of maximum curved surface area that can be inscribed in a given sphere of radius R.

Solution

Let h be the height of come and r be the radius of the cone. Consider the right ΔOMC where O is the centre of sphere and AM is perpendicular to the base BC of cone.

OM = h – R, OC = R, MC = r

$R^2 = (h - R)^2 + r^2$ ....(i)

and $r^2 + h^2 = \ell^2$ ....(ii)

where ℓ is the slant height of cone.

Curved surface area = C = $\pi r \ell$

Using (i) and (ii), express C in terms of h only.

$$C = \pi r \sqrt{r^2 + h^2}$$

$$\Rightarrow \quad C = \pi \sqrt{2hR - h^2} \ \sqrt{2hR}$$

We will maximise C^2.

Let $C^2 = f(h) = 2\pi^2 hR (2hR - h^2)$

$\Rightarrow \quad f'(h) = 2\pi^2 R (4hR - 3h^2)$

$f'(h) = 0 \quad \Rightarrow \quad 4hR - 3h^2 = 0$

$\Rightarrow \quad h = 4R/3.$

$f''(h) = 2\pi^3 R (4R - 6h)$

$$f''\left(\frac{4R}{3}\right) = 2\pi R^2 (4R - 8R) < 0$$

Hence curved surface area is maximum for $h = \frac{4R}{3}$

Using (i), we get $\quad r^2 = 2hR - h^2 = \frac{8R^2}{9}$

$$\Rightarrow \quad r = \frac{2\sqrt{2}}{3} R$$

Semi-vertical angle = $\theta \tan^{-1} r/h = \tan^{-1} 1/\sqrt{2}$

Example : 27

A cone is circumscribed about a sphere of radius R. Show that the volume of the cone is minimum if its height is 4R.

Solution

Let r be the radius, h be the height, and be the slant height of cone.

If O be the centre of sphere,

$\Delta AON - \Delta ACM$

$\Rightarrow \quad \frac{h-R}{R} = \frac{\ell}{r}$ ....(i)

$\Rightarrow \quad \frac{h-R}{R} = \frac{\sqrt{r^2+h^2}}{r}$

Squaring and simplifying we get ;

$r^2 = \frac{hR^2}{h-2R}$ ....(ii)

Now volume of cone = 1/3 πr^2 h

$\Rightarrow \quad V = \frac{1}{3}\pi\left(\frac{hR^2}{h-2R}\right)h$

$\Rightarrow \quad V = \frac{1}{3}\frac{\pi R^2}{\left(\frac{1}{h}-\frac{2R}{h^2}\right)}$

For volume to be minimum, the denominator should be maximum. Hence we will maximise :

$f(h) = \frac{1}{h} - \frac{2R}{h^2}$

$f'(h) = -\frac{1}{h^2} + \frac{4R}{h^3}$

$f'(h) = 0 \quad \Rightarrow \quad h = 4R$

$f''(h) = \frac{2}{h^3} - \frac{12R}{h^4} = \frac{2h-12R}{h^4}$

$f''(4R) = \frac{8R-12R}{256R^4} < 0$

Hence f(h) is maximum and volume is minimum for h = 4R.

Example : 28

The lower corner of a page in a book is folded over so as to reach the inner edge of the page. Show that the fraction of the width folded over when the area of the folded part is minimum is 2/3.

Solution

The corner A is folded to reach A_1.

The length of the folded part = AB = A_1B = x

Let total width = 1 unit

$\Rightarrow$ Length of the unfolded part = OB = 1 – x.

If CM || OA, $\Delta A_1CM \sim \Delta BA_1O$

$\Rightarrow \quad \frac{A_1C}{CM} = \frac{BA_1}{A_1O}$

$\Rightarrow \quad A_1C = y = CM\left(\frac{BA_1}{A_1O}\right)$

$\Rightarrow \quad y = 1\left(\frac{x}{\sqrt{x^2-(1-x)^2}}\right)$ ....(i)

Area of folded part = Area (ΔA_1BC)

$A = \frac{1}{2}\,xy = \frac{1}{2} \times \frac{x}{\sqrt{2x-1}}$

$\Rightarrow \quad A^2 = \frac{x^4}{4(2x-1)} = \frac{1}{4\left(\frac{2}{x^3}-\frac{1}{x^4}\right)}$

For area to be minimum, denominator in R.H.S. must be maximum.

Let $f(x) = \frac{2}{x^3} - \frac{1}{x^4}$

$f'(x) = \frac{-6}{x^4} + \frac{4}{x^5}$

$f'(x) = 0 \quad \Rightarrow \quad -6x + 4 = 0 \quad \Rightarrow \quad x = 2/3$

$f''(x) = \frac{24}{x^5} - \frac{20}{x^6} = \frac{24x-20}{x^6}$

$f''(2/3) = \frac{16-20}{(2/3)^6} < 0$

Hence f(x) is maximum and area is minimum if x = 2/3

i.e. 2/3 rd of the width

Example : 29

Prove that the minimum intercept made by axes on the tangent to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is a + b. Also find the ratio in which the point of contact divides this intercept.

Solution

Intercept made by the axes on the tangent is the length of the portion of the tangent intercepted between the axes. Consider a point P on the ellipse whose coordinates are x = a cost, y = b sin t
(where t is the parameter)

The equation of the tangent is :

$$y - b\sin t = \frac{b\cos t}{-a\sin t}(x - a\cos t)$$

$\Rightarrow \quad \frac{x}{a}\cos t + \frac{y}{b}\sin t = 1$

$\Rightarrow \quad OA = \frac{a}{\cos t},\ OB = \frac{b}{\sin t}$

Length of intercept = ℓ = AB = $\sqrt{\frac{a^2}{\cos^2 t}+\frac{b^2}{\sin^2 t}}$

We will minimise ℓ^2 .

Let $\ell^2 = f(t) = a^2\sec^2 t + \text{cosec}^2 t$

$\Rightarrow \quad f(t) - 2a^2\sec^2 t\tan t - 2b^2\text{cosec}^2 t\cos t$

$f'(t) = 0 \quad \Rightarrow \quad a^2\sin^4 t = b^2\cos^4 t$

$\Rightarrow \quad t = \tan^{-1}\sqrt{\frac{b}{a}}$

$f''(t) = 2a^2 (\sec^4 t + 2 \tan^2 t \sec^2 t) + 2b^2 (\text{cosec}^4 t + 2 \text{cosec}^2 t \cot^2 t)$
which is positive

Hence f(t) is minimum for tan t = $\sqrt{b/a}$

$\Rightarrow \quad \ell_{min} = \sqrt{a^2(1+b/a)+b^2(1+a/b)}$

$\Rightarrow \quad \ell_{min} = a + b \qquad$ ....(i)

$PA^2 = \left(a\cos t - \dfrac{a}{\cos t}\right)^2 + b^2 \sin^2 t = \dfrac{a^2 \sin^4 t}{\cos^2 t} + b^2 \sin^2 t = (a^2 \tan^2 t \;\; b^2) \sin^2 t = (ab + b^2) \dfrac{b}{a+b} = b^2$

$\Rightarrow \quad PA = b$

Using (i), PB = a

Hence $\dfrac{PA}{PB} = \dfrac{b}{a}$

$\Rightarrow$ P divides AB in the ratio b : a

Example : 30

Find the area of the greatest isosceles triangle that can be inscribed in a given ellipse having its vertex coincident with one end of the major axis.

Solution

Let the coordinates of B be (a cos t, b sin t)

$\Rightarrow$ The coordinates of C are : (a cos t, – b sin t)

because BC is a vertical line and BM = MC

Area of triangle = 1/2 (BC) (AM)

$\Rightarrow \quad A = 1/2 (2b \sin t) (a - a \cos t)$

$\Rightarrow \quad A(t) = ab (\sin t - \sin t \cos t)$

$A'(t) = ab (\cos t - \cos 2t)$

$A'(t) = ab (\cos t - \cos 2t)$

$A'(t) = 0 \quad \Rightarrow \quad \cos t - \cos 2t = 0$

$\Rightarrow \quad \cos t + 1 - 2 \cos^2 t = 0$

$\Rightarrow \quad \cos t = 1, -1/2$

$A''(t) = ab (-\sin t + 2 \sin 2t) = ab \sin t(4 \cos t - 1)$

$A''(2\pi/3) = ab \sqrt{3}/2 \; (-2 - 1) < 0$

Hence area is maximum for $t = \dfrac{2\pi}{3}$

Maximum area = $A\left(\dfrac{2\pi}{3}\right)$

$= ab (\sin 2\pi/3 - \sin 2\pi/3 \cos 2\pi/3)$

$= ab\left(\dfrac{\sqrt{3}}{2} + \dfrac{\sqrt{3}}{2}\dfrac{1}{2}\right) = \dfrac{3\sqrt{3}}{4} ab$

Example : 31

Find the point on the curve $y = x^2$ which is closest to the point A (0, a)

Solution

Using the parametric representation, consider an arbitrary point P (t, t^2) on the curve.

Distance of P from A = PA

$PA = \sqrt{t^2 + (t^2 - a)^2}$

We have to find t so that this distance is minimum.

We will minimise PA^2

Let $PA^2 = f(t) = t^2 + (t^2 - a)^2$

$f'(t) = 2t + 4t (t^2 - a)$

$f'(t) = 2t\,[2t^2 - 2a + 1]$

$f'(t) = 0 \qquad \Rightarrow \qquad t = 0, \pm\sqrt{a - \frac{1}{2}}$

$f''(t) = 2 - 4a + 12\,t^2$

we have to consider two possibilities.

Case – I : $a < 1/2$

In this case, $t = 0$ is the only value.

$f''(0) = 2 - 4a = 4\,(1/2 - a) > 0$

Hence the closest point corresponds to $t = 0$

$\Rightarrow$ (0, 0) is the closest point

Case – II : $a > 1/2$

In this case $t = 0, \pm\sqrt{a - \frac{1}{2}}$

$f''(0) = 2 - 4a = 4\left(\frac{1}{2} - a\right) < 0$

$\Rightarrow$ local maximum at $t = 0$

$f''\left(\pm\sqrt{a - \frac{1}{2}}\right) = 2 - 4a + 12\,a - 6 = 8\left(a - \frac{1}{2}\right) > 0$

Hence the distance is minimum for $t = \pm\sqrt{a - \frac{1}{2}}$

So the closest points are $\left(\sqrt{a - \frac{1}{2}}, \frac{2a - 1}{2}\right)$ and $\left(-\sqrt{a - \frac{1}{2}}, \frac{2a - 1}{2}\right)$

Example : 32

Find the shortest distance between the line $y - x = 1$ and the curve $x = y^2$

Solution

Let P (t^2, t) be any point on the curve $x = y^2$. The distance of P from the given line is = $\frac{|-t^2 + t - 1|}{\sqrt{t^2 + 1^2}}$

$= \frac{t^2 - t + 1}{\sqrt{2}}$ because $t^2 - t + 1$ is a positive expression. We have to find minimum value of this expression.

Let $f(t) = t^2 - t + 1$

$f'(t) = 2t - 1$

$f'(t) = 0 \qquad \Rightarrow \qquad t = 1/2$

$f''(t) = 2 > 0$

$\Rightarrow$ distance is minimum for $t = +1/2$

Shortest distance = $\left[\frac{t^2 - t + 1}{\sqrt{2}}\right]_{t=1/2} = \frac{\frac{1}{4} - \frac{1}{2} + 1}{\sqrt{2}} = \frac{3\sqrt{2}}{8}$

Example : 33

Find the point on the curve $4x^2 + a^2y^2 = 4a^2$; $4 < a^2 < 8$ that is farthest from the point (0, –2).

Solution

The given curve is an ellipse $\frac{x^2}{a^2} \quad \frac{y^2}{4} = 1$

Consider a point (a cost, 2 sin t) lying on this ellipse.

The distance of P from (0, –2) = $\sqrt{a^2\cos^2 t + (2 + 2\sin t)^2}$

This distance is to be maximised.

Let $f(t) = a^2 \cos^2 t + 4(1 + \sin t)^2$

$f'(t) = -2a^2 \sin t \cos t + 8(1 + \sin t)(\cos t)$

$f'(t) = (8 - 2a^2) \sin t \cos t + 8 \cos t$

$f'(t) = 0 \Rightarrow \quad \cos t = 0 \quad$ or $\quad \sin t = \frac{4}{a^2 - 4}$

$\Rightarrow \quad t = \pi/2$ or $t = \sin^{-1}\left(\frac{4}{a^2 - 4}\right)$

($t = 3\pi/2$ is rejected because it makes the distance zero)

Let us first discuss the possibility of $t = \sin^{-1}\left(\frac{4}{a^2 - 4}\right)$

We are give that $4 < a^2 < 8$

$\Rightarrow \quad 0 < a^2 - 4 < 4$

$\Rightarrow \quad 0 < 1 < \frac{4}{a^2 - 4}$

as $\frac{4}{a^2 - 4}$ is greatest than 1,

$t = \sin^{-1}\frac{4}{a^2 - 4}$ is not possible.

Hence $t = \pi/2$ is the only value.

Now, $f''(t) = (8 - 2a^2) \cos 2t - 8 \sin t$

$f''(\pi/2) = 2a^2 - 8 - 8 = 2(a^2 - 8) < 0$

$\Rightarrow$ The farthest point corresponds to $t = \pi/2$ and its

Coordinates are $\equiv (a \cos \pi/2, 2 \sin \pi/2) \equiv (0, 2)$

Example : 34

If a + b + c = 0, then show that the quadratic equation $3ax^2 + 2bx + c = 0$, has at least one root in 0 and 1.

Solution

Consider the polynomial $f(x) = ax^3 + bx^2 + cx$. We have f(0) = 0 and f(1) = a + b + c = 0 (Given)

$\Rightarrow \quad f(0) = f(1)$

Also f(x) is continuous and differentiable in [0, 1], it means Rolle's theorem is applicable.

Using the Rolle's Theorem there exists a root of $f'(x) = 0$

i.e. $\quad 3ax^2 + 2bx + c = 0$ between 0 and 1

Hence proved.

Example : 35

Let A (x_1, y_1) and B(x_2, y_2) be any two points on the parabola $y = ax^2 + bx + c$ and let C (x_3, y_3) be the point on the arc AB where the tangent is parallel to the chord AB. Show that $x_3 = (x_1 + x_2)/2$.

Solution

Clearly $f(x) = ax^2 + bx + c$ is a continuous and differentiable function for all values of $x \in [x_1, x_2]$.

On applying Langurange's Mean value theorem on f(x) in (x_1, x_2) we get

$$f'(x_3) = \frac{f(x_2) - f(x_1)}{x_2 - x_1} \qquad [\because \quad x_3 \in (x_1, x_2)]$$

On differentiating f(x), we get :

$f'(x) = 2ax + b \quad \Rightarrow \quad f'(x_3) = 2ax_3 + b$

On substituting x_1 and x_2 in the quadratic polynomial, we get

$f(x_1) = ax_1^2 + bx_1 + c$ and $f(x_2) = ax_2^2 + bx_2 + c$

On substituting the values of $f(x_1)$, $f(x_2)$ and $f'(x_3)$ in (i), we get :

$$2ax_3 + b = \frac{ax_2^2 + bx_2 + c - (ax_1^2 + bx_1 + c)}{x_2 - x_1}$$

$\Rightarrow \quad ax_3 = a(x_1 + x_3)$

$\Rightarrow \quad x_3 = \frac{x_1 + x_2}{2}$. Hence Proved

Example : 36

Find the condition so that the line ax + by = 1 may be a normal to the curve $a^{n-1} y = x^n$.

Solution

Let (x_1, y_1) be the point of intersection of line ax + by = 1 and curve $a^{n-1} y = x^n$.

$\Rightarrow \quad ax_1 + by_1 = 1$ ....(i)

and $\quad a^{n-1} y_1 = x_1^n$ ....(ii)

The given curve is : $a^{n-1}y = x^n$

$$\Rightarrow \quad \left.\frac{dy}{dx}\right]_{at(x_1, y_1)} = n\,\frac{x_1^{n-1}}{a^{n-1}} = n\,\frac{x_1^{n-1}}{x_1^n}\,y_1 = \frac{ny_1}{x_1} \qquad \text{[using (ii)]}$$

Equation of normal to (x_1, y_1) is :

normal is $y - y_1 = \frac{-x_1}{ny_1}(x - x_1)$

$\Rightarrow \quad xx_1 + ny\,y_1 = ny_1^2 + x_1^2$ .... (iii)

But the normal is the line xa + yb = 1 .... (iv)

Comparing (iii) and (iv), we get

$$\frac{x_1}{a} = \frac{ny_1}{b} = \frac{ny_1^2 + x_1^2}{1}$$

Let each of these quantities by K, i.e. $\frac{x_1}{a} = \frac{ny_1}{b} = \frac{ny_1^2 + x_1^2}{1} = K$

$\Rightarrow \quad x_1 = aK,\ ny_1 = bK,\ ny_1^2 + x_1^2 = K$

On substituting the values of x_1 and y_1 from first two equations into third equation, we get

$$n\,\frac{b^2K^2}{n^2} + a^2K^2 = K$$

$$\Rightarrow \quad K = \frac{n}{b^2 + na^2},\ x_1 = \frac{an}{b^2 + na^2} \text{ and } y_1 = \frac{b}{b^2 + na^2}$$

Replacing the values of x_1 and y_1 in (ii), we get :

$$a^{n-1}\,\frac{b}{b^2 + na^2} = \left(\frac{an}{b^2 + na^2}\right)^n \text{ as the required condition.}$$

Example : 37

Find the vertical angle of right circular cone of minimum curved surface that circumscribes in a given sphere.

Solution

When cone is circumscribed over a sphere
we have : $\Delta AMC \sim \Delta APO$

$$\Rightarrow \quad \frac{AC}{MC} = \frac{AO}{OP} \Rightarrow \frac{\ell}{r} = \frac{r-R}{R} \qquad ....(i)$$

In cone, we can define $r^2 + h^2 = \ell^2$ ....(ii)
Eliminating ℓ in (i) and (ii), we get

$$r^2 = \frac{hR^2}{h-2R} \qquad ....(iii)$$

Let curved surface area of cone = $C - \pi r \ell$

$$\Rightarrow \quad C = \pi r \, \frac{r(h-R)}{R} \qquad \text{[using (i)]}$$

$$\Rightarrow \quad C = \frac{\pi h R(h-R)}{(h-2R)} \qquad \text{[using (iii)]}$$

As C is expressed in terms on one variable only i.e. h, we can maximise C by use of derivatives

$$\frac{dC}{dh} = \frac{\pi R}{(h-2R)^2}\,[(h-2R)(2h-R) - (h^2 - hR)] = 0$$

$$\Rightarrow \quad h^2 - 4Rh + 2R^2 = 0$$

$$\Rightarrow \quad h = (2 + \sqrt{2})\,R \qquad ....(iv)$$

It can be shown that $\frac{d^2C}{dh^2} > 0$ for this value of h.

Substituting $h = (2 + \sqrt{2})\,R$ in (iii), we get $\qquad \frac{(\sqrt{2}+1)\,R^2}{(\sqrt{2}+1)\,R^2 + 2(\sqrt{2}+1)^2\,R^2}$

$r^2 = (\sqrt{2} + 1)\,R^2$

Let semi-vartical angle = θ

$$\Rightarrow \quad \sin^2\theta = r^2/\ell^2 = \frac{r^2}{r^2 + h^2}$$

Using (iv) and (v), we get :

$$\sin^2\theta = \qquad\qquad = \frac{1}{3 + 2\sqrt{2}}$$

$$\Rightarrow \quad \sin^2\theta = 3 - 2\sqrt{2} = (\sqrt{2} - 1)^2$$

$$\Rightarrow \quad \sin\theta = \sqrt{2} - 1$$

Example : 38

Let $f(x) = x^3 - x^2 + x + 1$ and $g(x) = \begin{cases} \max[f(t)] & 0 \le t \le x \quad 0 \le x \le 1 \\ 3 - x & ; \quad 1 < x \le 2 \end{cases}$

Discuss the continuity and differentiability of f(x) in (0, 2)

Solution

It is given that $f(x) = x^3 - x^2 + x + 1$
$f'(x) = 3x^2 - 2x + 1$
$f'(x) > 0$ for all x
($\because$ coeff. of $x^2 > 0$ and Discriminant < 0)
Hence f(x) is always increasing function.
Consider $0 \le t \le x$

$\Rightarrow$ f(0) ≤ f(t) ≤ f(x) (∵ f(t) is an in creasing function)

$\Rightarrow$ 1 ≤ f(t) ≤ f(x)

$\Rightarrow$ Maximum [f(t)] = f(x) = $x^3 - x^2 + x + 1$

$$\Rightarrow \quad g(x) = \begin{cases} x^3 - x^2 + x + x + 1 & , \quad 0 \le x \le 1 \\ 3 - x & , \quad 1 < x \le 2 \end{cases}$$

As g(x) is polynomial in [0, 1] and (1, 2], it is continuous and differentiable in these intervals.

At x = 1

LHL = 2, RHL = 2 and f(1) = 2

$\Rightarrow$ g(x) is continuous at x = 1

LHD = 2 and RHD = – 1

$\Rightarrow$ g(x) is non-differentiable at x = 1

Example : 39

Two considers of width a and b meet at right angles show that the length of the longest pipe that can be passes round the corner horizontally is $(a^{2/3} + b^{2/3})^{3/2}$

Solution

Consider a segment AB touching the corner at P. AB = a cosec θ + b sec θ

Let f(θ) = a cosec θ + b sec θ ....(i)

f′(θ) = – a cosec θ cot θ + b sec θ tan θ

$$\Rightarrow \quad f'(\theta) = \frac{-a\cos\theta}{\sin^2\theta} + \frac{b\sin\theta}{\cos^2\theta} = \frac{-a\cos^3\theta + b\sin^3\theta}{\sin^2\theta\cos^2\theta}$$

f′(θ) = 0 $\Rightarrow$ $\tan^3\theta = a/b$

$\Rightarrow$ $\tan\theta = (a/b)^{1/3}$

Using first derivative test, see yourself that f(θ) possesses local minimum at $\theta = \tan^{-1}(a/b)^{1/3}$.

Using (i), the minimum length of segment AB is :

$$f_{min} = (a^{2/3} + b^{2/3})^{3/2} \quad \text{for } \theta = \tan^{-1}\sqrt[3]{\frac{b}{a}}$$

This is the minimum length of all the line segments that can be drawn through corner P. If the pipe passes through this segment, it will not get blocked in any other position. Hence the minimum length of segment APB gives the maximum length of pipe that can be passed.

Example : 40

Find the equation of the normal to the curve $y = (1 + x)^y + \sin^{-1}(\sin^2 x)$ at x = 0.

Solution

We have

$y = (1 + x)^y + \sin^{-1}(\sin^2 x)$

Let $A = (1 + x)^y$ and $B = \sin^{-1}\sin^2 x$

$\Rightarrow$ y = A + B ....(i)

Consider A

Taking log and differentiating, we get

$\ell n\, A = y\, \ell n\,(1 + x)$

$$\frac{1}{A}\frac{dA}{dx} = \frac{dy}{dx}\,\ell n\,(1 + x) + \left(\frac{y}{1+x}\right)$$

$$\text{or} \quad \frac{dA}{dx} = A\left[\frac{dy}{dx}\ell n(1+x) + \frac{y}{1+x}\right] = (1 + x)^y\left[\frac{dy}{dx}\ell n(1+x) + \frac{y}{1+x}\right] \quad ....(ii)$$

Consider B

$B = \sin^{-1}(\sin^2 x)$ $\Rightarrow$ $\sin B = \sin^2 x$

Differentiating wrt x, we get

$$\cos B\,\frac{dB}{dx} = 2\sin x\cos x$$

$$\frac{dB}{dx} = \frac{1}{\cos B} \ (2 \sin x \cos x) = \frac{2\sin x \cos x}{(1-\sin^2 B)^{1/2}} = \frac{2\sin x \cos x}{(1-\sin^4 x)^{1/2}}$$

Now since $\quad y = A + B$

we have $\quad \frac{dy}{dx} = \frac{dA}{dx} + \frac{dB}{dx} = (1 + x)^y \left[\frac{dy}{dx}\ \ell n\,(1+x) + \frac{y}{1+x}\right] + \frac{2\sin x \cos x}{(1-\sin^2 x)^{1/2}}$

or $\quad \frac{dy}{dx} = \frac{y(1+x)^{y-1} + \frac{2\sin x \cos x}{(1-\sin^4 x)^{1/2}}}{1-(1-x)^y\,\ell n\,(1+x)}$

Using the equation of given curve, we can find f(0).

Put x = 0 in the given curve.

$y = (1 + 0)^y + \sin^{-1}(\sin^2 0) = 1$

$$\frac{dy}{dx} = \frac{1(1+0)^{1-1} + \frac{2\sin 0 \cos 0}{(1-\sin^4 0)^{1/2}}}{1-(1-0)^1\,\ell n(1+0)} \quad \Rightarrow \quad \frac{dy}{dx} = 1$$

The slope of the normal is $m = -\frac{1}{(dy/dx)} = -1$

Thus, the required equation of the normal is y – 1 = (–1) (x – 0)

i.e. $\quad y + x - 1 = 0$

Example : 41

Tangent at a point P_1(other than (0, 0) on the curve $y = x^3$ meets the curve again at P_2. The tangent at P_2 meets the curve at P_3, and so on. Show that the abscissa of P_1, P_2, P_3 ......P_n , from a GP. Also find the ratio [area ($\Delta P_1 P_2 P_3$)] / [area ($\Delta P_2 P_3 P_4$)].

Solution

Let the chosen point on the curve $y = x^3$ be P_1 (t, t^3). The slope of the tangent to the curve at (t, t^3) is

given as $\frac{dy}{dx} = 3x^2 = 3t^2$ ....(i)

The equation of the tangent at (t, t^3) is

$y - t^3 = 3t^2 (x - t)$

$y - 3t^2 x + 2t^3 = 0$ ....(ii)

Now to get the points where the tangent meets the curve again, solve their equations

i.e. $\quad x^3 - 3t^2 x + 2t^3 = 0$ ....(iii)

One of the roots of this equation must be the sbscissa of P_1 i.e. t. Hence, equation (iii) can be factorised as

$(x - t)(x^2 + tx - 2t^2) = 0$

or $\quad (x - t)(x - t)(x + 2t) = 0$

or $\quad (x - t)(x - t)(x + 2t) = 0$

Hence, the abscissa of $P_2 = -2t$ ....(iv)

Let coordinates of point P_2 are (t_1, t_1^3)

Equation of tangent at P_2 is : $y - 3t_1^2 x + 2t_1^3 = 0$

[this is written by replacing t by t_1 in (ii)]

On solving tangent at P_2 and the given curve we get the coordinates of the point where tangent at P_2 meets the curve again i.e.

coordinates of P_3 are $(-2t_1, -t_1)$

Using (iv), abscissa of $P_3 = -2(-2)t$

$\Rightarrow \quad$ abscissa of $P_3 = 4t$

So the abscissa of P_1, P_2 and P_3 are t, (–2) t, (–2) (–2) t respectively, that is, each differing from the preceding one by a factor of (–2).

Hence, we conclude that the abscissae of P_1, P_2, P_3, ..., P_n from a GP with common ratio of – 2.

Now area $(\Delta P_1 P_2 P_3) = \frac{1}{2}\begin{vmatrix} t & t^3 & 1 \\ -2t & -8t^3 & 1 \\ 4t & 64t^3 & 1 \end{vmatrix} = \frac{t^4}{2}\begin{vmatrix} 1 & 1 & 1 \\ -2 & -8 & 1 \\ 4 & 64 & 1 \end{vmatrix}$

area $(\Delta P_2 P_3 P_4) = \frac{1}{2}\begin{vmatrix} -2t & -8t^3 & 1 \\ 4t & 64t^3 & 1 \\ -8t & (-2)^9 t^3 & 1 \end{vmatrix} = \frac{16t^4}{2}\begin{vmatrix} 1 & 1 & 1 \\ -2 & -8 & 1 \\ 4 & 64 & 1 \end{vmatrix}$

Hence $\frac{\text{area}(\Delta P_1 P_2 P_3)}{\text{area}(\Delta P_2 P_3 P_4)} = \frac{t^4}{16t^4} = \frac{1}{16}$

Example : 42

Let $f(x) = \begin{cases} -x^3 + \frac{(b^3 - b^2 + b - 1)}{(b^2 + 3b + 2)} & , \quad 0 \le x < 1 \\ 2x - 3 & , \quad 1 \le x \le 3 \end{cases}$

Find all possible real values of b such that f(x) has the smallest value at x = 1

Solution

The value of function f(x) at x = 1 is f(x) = 2x – 3 = 2 (1) – 3 = – 1

The function f(x) = 2x – 3 is an increasing function on [1, 3]. hence, f(1)= – 1 is the smallest value of f(x) at x = 1.

Now $\quad f(x) = -x^3 + \frac{(b^3 - b^2 + b - 1)}{(b^2 + 3b + 2)}$

is a decreasing function on [0, 1] for fixed values of b. So its smallest value will occur at the right end of the interval.

$\Rightarrow$ Minimum [(f(x) in [0, 1]) ≥ -1

$\Rightarrow$ $f(1) \ge -1$

$-1 + \frac{(b^3 - b^2 + b - 1)}{(b^2 + 3b + 2)} \ge -1$

In order that this value is not less than – 1, we must have $\frac{b^3 - b^2 + b - 1}{b^2 + 3b + 2} \ge 0$

$\Rightarrow \quad \frac{(b^2 + 1)(b - 1)}{(b + 2)(b + 1)} \ge 0 \qquad \Rightarrow \quad \frac{(b - 1)}{(b + 2)(b + 1)} \ge 0$

The sign of b is positive for $b \in (-2, -1) \cup [1, \infty)$

Hence, the possible real values of b such that f(x) has the smallest value at x = 1 are $(-2, -1) \cup [1, \infty)$

Example : 43

Find the locus of a point that divides a chord of slope 2 of the parabola $y^2 = 4x$ internally in the ratio 1 : 2.

Solution

Let $P \equiv (t_1^2, 2t_1)$, $Q \equiv (t_2^2, 2t_2)$ be the end points of chord AB. Also let $M \equiv (x_1, y_1)$ be a point which divides AB internally in ratio 1 : 2.

It is given that slope of PQ = 2,

$\Rightarrow \quad$ slope (PQ) = $\frac{2t_2 - 2t_1}{t_2^2 - t_1^2} = 2$

$\Rightarrow \quad t_1 + t_2 = 1$ ....(i)

As M divides PQ in 1 : 2 ratio, we get

$\Rightarrow \quad x_1 = \frac{2t_1^2 + t_2^2}{3}$ ....(ii) and $y_1 = \dfrac{2t_2 + 4t_1}{3}$ ....(iii)

We have to eliminate two variables t_1 and t_2 between (i), (ii) and (iii).
From (i), put $t_1 = 1 – t_2$ in (iii) to get :
$3y_1 = 2 (t – t_2) + 4t_2 = 2 (1 + t_2)$
$\Rightarrow$ $t_2 = (3y – 2)/2$ and $t_1 = –3y_1/2$
On substituting the values of t_1 and t_2 in (ii), we get : $4x_1 = 9y_1^2 – 16y_1 + 8$
Replacing x_1 by x and y_1 by y, we get the required locus as : $4x = 9y^2 – 16y + 8$

Example : 44

Determine the points of maxima and minima of the function $f(x) = 1/8\ \ell n\ x – bx + x^2 + x^2$, $x > 0$, where $b \ge 0$ is a constant.

Solution

Consider $f(x) = 1/8\ \ell n\ x – bx + x^2$
$\Rightarrow$ $f'(x) = 1/8\ x – b + 2x = 0$
$\Rightarrow$ $16x^2 – 8bx + 1 = 0$

$\Rightarrow$ $x = \dfrac{b \pm \sqrt{b^2 - 1}}{4}$

For $0 \le b < 1$ $f'(x) > 0$ for all x
$\Rightarrow$ f(x) is an increasing function
$\Rightarrow$ No local maximum or local minimum

For $b > 1$ $f'(x) = 0$ at $x_1 = \dfrac{b - \sqrt{b^2 - 1}}{4}$ and $x_2 = \dfrac{b + \sqrt{b^2 - 1}}{4}$

Check yourself that x_1 is a point of local maximum and x_2 is a point of local minimum.
For b = 1
$f'(x) = 16x^2 – 8x^2 + 1 = (4x – 1)^2 = 0$
$\Rightarrow$ $x = 1/4$
$f''(x) = 2 (4x – 1) (4)$
$\Rightarrow$ $f''(1/4) = 0$
$f'''(x) = 32$ $\Rightarrow$ $f'''(1/4) \ne 0$
$\Rightarrow$ 1/4 is a point of inflexion
i.e. no local maxima or minima
So points of local maximum and minimum are :
$0 \le b \le 1$: No local maximum or minimum

$b > 1$: Local maximum at $x = \dfrac{b - \sqrt{b^2 - 1}}{4}$

Local minimum at $x – \dfrac{b + \sqrt{b^2 - 1}}{4}$

Example : 45

Let $f(x) = \begin{cases} xe^{ax} & , \quad x \le 0 \\ x + ax^2 - x^3 & , \quad x > 0 \end{cases}$

Where a is a positive constant. Find the interval in which $f'(x)$ is increasing

Solution

Consider $x \le 0$
$f'(x) = e^{ax} (1 + xa)$
$f''(x) = a\ e^{ax} (1 + xa) + e^{ax}\ a$
$\Rightarrow$ $f''(x) = e^{ax} (2a + xa^2) > 0$
$\Rightarrow$ $x > – 2/a$ ($\because$ e^{ax} is always + ve)
So $f'(x)$ in increasing in $–2/a < x < 0$ ....(i)
Consider $x > 0$

$f'(x) = 1 + 2ax - 3x^2$

$f''(x) = 2a - 6x > 0 \quad \Rightarrow \quad x < a/3$

$\Rightarrow \quad f'(x)$ is increasing in $0 < x < a/3$ ....(ii)

From (i) and (ii), we can conclude that :

$f'(x)$ is increasing in $x \in (-2/a, 0) \cup (0, a/3)$

Example : 46

What normal to the curve $y = x^2$ forms the shortest chord?

Solution

Let (t, t^2) be any point P on the parabola $y = x^2$

Equation of normal at P to $y = x^2$ is :

$y - t^2 = -1/2t\,(x - t)$

Now assume that normal at P meets the curve again at Q whose coordinates are (t_1, t_1^2).

$\Rightarrow$ The point $Q(t_1, t_1^2)$ should satisfy equation of the normal

$\Rightarrow \quad t_1^2 - t^2 - 1/2t\,(t_1 - t)$

$\Rightarrow \quad t_1 + t = -1/2t \quad \Rightarrow \quad t_1 = -t - 1/2t$ ....(i)

$PQ^2 = (t - t_1)^2 + (t^2 - t_1^2)^2 = (t - r_1)^2\ [1 + (t_1 + t)^2]$

On substituting the value of t_1 from (i), we get ;

$$\Rightarrow \quad PQ^2 = \left(2t + \frac{1}{2t}\right)^2 \left(1 + \frac{1}{4t^2}\right) = 4t^2 \left(1 + \frac{1}{4t^2}\right)^3$$

Let $PQ^2 = f(t)$

$$\Rightarrow \quad f'(f) = 8t\left(1 + \frac{1}{4t^2}\right)^3 + 12t^2\left(1 + \frac{1}{4t^2}\right)^2 \left(\frac{-2}{4t^3}\right)$$

$$\Rightarrow \quad f'(t) = 2\left(1 + \frac{1}{4t^2}\right)^2 \left[4t\left(1 + \frac{1}{4t^2}\right) - \frac{3}{t}\right]$$

$f'(t) = 0 \quad \Rightarrow \quad 2t - 1/t = 0$

$\Rightarrow \quad t^2 = 1/2 \quad \Rightarrow \quad t = \pm 1/\sqrt{2}$

It is easy to see $f''(t) > 0$ for $t = \pm 1/\sqrt{2}$

equation of PQ :

for $t = 1\sqrt{2} \equiv \sqrt{2}\,x + 2y - 2 = 0$ and

for $t = -1\sqrt{2} \equiv \sqrt{2}\,x - 2y + 2 = 0$