

**Example : 1**

Let  $A = \{1, 2, 3\}$  and  $B = \{4, 5\}$ . Check whether the following subsets of  $A \times B$  are functions from  $A$  to  $B$  or not.

- (i)  $f_1 = \{(1, 4), (1, 5), (2, 4), (3, 5)\}$                       (ii)  $f_2 = \{(1, 4), (2, 4), (3, 4)\}$   
 (iii)  $f_3 = \{(1, 4), (2, 5), (3, 5)\}$                                       (iv)  $f_4 = \{(1, 4), (2, 5)\}$

**Solution**

- (i)  $f_1 = \{(1, 4), (1, 5), (2, 4), (3, 5)\}$   
 It is not a function since an element of domain (i.e. 1) has two image in co-domain (i.e. 4, 5)
- (ii)  $f_2 = \{(1, 4), (2, 4), (3, 4)\}$   
 It is function as every element of domain has exactly one image  $f(A) = \text{Range} = \{4\}$
- (iii)  $f_3 = \{(1, 4), (2, 5), (3, 5)\}$   
 It is a function.  $f(A) = \text{Range} = \{4, 5\} = \text{co-domain}$
- (iv)  $f_4 = \{(1, 4), (2, 5)\}$   
 It is not a function because one element (i.e. 3) in domain does not have an image

**Example : 2**

Which of the following is a function from  $A$  to  $B$ ?

- (i)  $A = \{x \mid x > 0 \text{ and } x \in \mathbb{R}\}$ ,  $B = \{y \mid y \in \mathbb{R}\}$   
 ( $A$  is the set of positive real numbers and  $B$  is the set of all real numbers)  
 $f = \{(x, y) \mid y = \sqrt{x}\}$
- (ii)  $A = \{x \mid x \in \mathbb{R}\}$ ,  $B = \{y \mid y \in \mathbb{R}\}$   
 $f = \{(x, y) \mid y = \sqrt{x}\}$

**Solution**

- (i)  $f$  is a function from  $A$  to  $B$  because every element of domain (+ve reals) has a unique image (square root) in codomain
- (ii)  $f$  is not a function from  $A$  to  $B$  because –ve real nos. are present in domain and they do not have any image in codomain  
 ( $\because y = \sqrt{x}$  is meaningless for –ve reals of  $x$ )

**Example : 3**

Check the following functions for injective and surjective

- (i)  $f : \mathbb{R} \rightarrow \mathbb{R}$  and  $f(x) = x^2$   
 (ii)  $f : \mathbb{R} \rightarrow \mathbb{R}^+$  and  $f(x) = x^2$   
 (iii)  $f : \mathbb{R}^+ \rightarrow \mathbb{R}^+$  and  $f(x) = x^2$

**Solution**

- (i) Injective  
 Let  $f(x_1) = f(x_2) \Rightarrow x_1^2 = x_2^2 \Rightarrow x_1 = \pm x_2$   
 $\Rightarrow$  it is not necessary that  $x_1 = x_2$   
 $\Rightarrow$  It is not injective
- Surjective  
 $y = x^2$   
 $\Rightarrow x = \pm \sqrt{y}$  for –ve values of  $y$  in codomain, there does not exist any value of  $x$  in domain  
 $\Rightarrow$  It is not surjective
- (ii) Injective  
 Let  $f(x_1) = f(x_2) \Rightarrow x_1^2 = x_2^2 \Rightarrow x_1 = \pm x_2 \Rightarrow$  not injective
- Surjective  
 $y = x^2 \Rightarrow x = \pm \sqrt{y}$   
 As the codomain contains only positive real numbers, there exists some  $x$  for every values of  $y$   
 $\Rightarrow$  it is surjective
- (iii) Injective  
 $f(x_1) = f(x_2) \Rightarrow x_1^2 = x_2^2 \Rightarrow x_1 = x_2$  because domain contains only +ve reals  
 $\Rightarrow$  it is injective
- Surjective  
 $y = x^2 \Rightarrow x = \pm \sqrt{y}$   
 for +ve values of  $y$ , there exists some  $x$ , As codomain is  $\mathbb{R}^+$ , it is surjective

**Example : 4**

Let  $A = \mathbb{R} - \{3\}$  and  $B = \mathbb{R} - \{1\}$

Let  $f : A \rightarrow B$  be defined by  $f(x) = \frac{x-2}{x-3}$

Is  $f$  bijective ?

**Solution**

Injective

Let  $f(x_1) = f(x_2)$  where  $x_1, x_2 \in A$

$$\Rightarrow \frac{x_1-2}{x_1-3} = \frac{x_2-2}{x_2-3} \quad \Rightarrow \quad (x_1-2)(x_2-3) = (x_2-2)(x_1-3) \quad (\text{because } x_1, x_2 \neq 3)$$

$$\Rightarrow x_1 = x_2 \quad (\text{on simplification})$$

Hence  $f(x)$  is injective

Surjective

$$y = \frac{x-2}{x-3}$$

$$\Rightarrow y(x-3) = x-2$$

$$\Rightarrow x = \frac{3y-2}{y-1}$$

For  $y \neq 1$ , there exists some value of  $x$ . As the codomain does not contain 1, we have some value of  $x$  in domain for every value of  $y$  in codomain

$\Rightarrow$  it is surjective

Hence  $f(x)$  is bijective

Inverse of  $f(x)$

$$\text{Interchanging } x \text{ and } y \text{ in } y = f(x) \text{ we have } x = \frac{y-2}{y-3} \quad \Rightarrow \quad y = \frac{3x-2}{x-1}$$

$$\Rightarrow f^{-1}(x) = \frac{3x-2}{x-1} \text{ is the inverse of } f(x)$$

**Example : 5**

Is  $f : \mathbb{R} \rightarrow \mathbb{R}$ ,  $f(x) = \cos(5x+2)$  invertible ?

**Solution**

Injective

Let  $f(x_1) = f(x_2)$  where  $x_1, x_2 \in \mathbb{R}$

$$\Rightarrow \cos(5x_1+2) = \cos(5x_2+2)$$

$$\Rightarrow 5x_1+2 = 2n\pi \pm (5x_2+2)$$

$$\Rightarrow \text{it is not necessary that } x_1 = x_2$$

hence if it not injective

Surjective

$$y = \cos(5x+2)$$

$$\Rightarrow x = \frac{\cos^{-1}y - 2}{5}$$

$$\Rightarrow \text{there is no value of } x \text{ for } y \in (-\infty, -1) \cup (1, +\infty)$$

As this interval is included in codomain, there are some values of  $y$  in codomain for which there does no exist any value of  $x$ . Hence it is not surjective.

As  $f$  is neither injective nor surjective, it is not invertible.

**Example : 6**

- (i) Let  $f(x) = x - 1$  and  $g(x) = x^2 + 1$ .  
What is  $f \circ g$  and  $g \circ f$ ?
- (ii)  $f = \{(1, 2), (3, 5), (4, 1)\}$  and  $g = \{(2, 3), (5, 1), (1, 3)\}$   
write down the pairs in the mappings  $f \circ g$ .

**Solution**

- (i)  $f \circ g = f[g(x)] = f(x^2 + 1) = x^2 + 1 - 1 = x^2$   
 $g \circ f = g[f(x)] = g[x - 1] = (x - 1)^2 + 1$
- (ii) domain of  $f \circ g$  is the domain of  $g(x)$  i.e.  $\{2, 5, 1\}$   
 $f \circ g(2) = f[g(2)] = f(3) = 5$   
 $f \circ g(5) = f[g(5)] = f(1) = 2$   
 $f \circ g(1) = f[g(1)] = f(3) = 5$   
 $\Rightarrow f \circ g = \{(2, 5), (5, 2), (1, 5)\}$

**Example : 7**

If  $A = \left\{x : \frac{\pi}{6} \leq x \leq \frac{\pi}{3}\right\}$  and  $f(x) = \cos x - x(1 + x)$ . Find  $f(A)$ .

**Solution**

We have to find the range with  $A$  as domain.  
As  $f(x)$  is decreasing in the given domain

$$\frac{\pi}{6} \leq x \leq \frac{\pi}{3} \quad \Rightarrow \quad f\left(\frac{\pi}{6}\right) \geq f(x) \geq f\left(\frac{\pi}{3}\right)$$

$$\Rightarrow f(x) \in \left[ \frac{1}{2} - \frac{\pi}{3} - \frac{\pi^2}{9}, \frac{\sqrt{3}}{2} - \frac{\pi^2}{36} \right]$$

$$\Rightarrow \text{the range is the interval : } \left[ \frac{1}{2} - \frac{\pi}{3} - \frac{\pi^2}{9}, \frac{\sqrt{3}}{6} - \frac{\pi}{6} - \frac{\pi^2}{36} \right]$$