

Example : 1

Let $A = \{1, 2, 3\}$ and $B = \{4, 5\}$. Check whether the following subsets of $A \times B$ are functions from A to B or not.

- (i) $f_1 = \{(1, 4), (1, 5), (2, 4), (3, 5)\}$ (ii) $f_2 = \{(1, 4), (2, 4), (3, 4)\}$
 (iii) $f_3 = \{(1, 4), (2, 5), (3, 5)\}$ (iv) $f_4 = \{(1, 4), (2, 5)\}$

Solution

- (i) $f_1 = \{(1, 4), (1, 5), (2, 4), (3, 5)\}$
 It is not a function since an element of domain (i.e. 1) has two image in co-domain (i.e. 4, 5)
- (ii) $f_2 = \{(1, 4), (2, 4), (3, 4)\}$
 It is function as every element of domain has exactly one image $f(A) = \text{Range} = \{4\}$
- (iii) $f_3 = \{(1, 4), (2, 5), (3, 5)\}$
 It is a function. $f(A) = \text{Range} = \{4, 5\} = \text{co-domain}$
- (iv) $f_4 = \{(1, 4), (2, 5)\}$
 It is not a function because one element (i.e. 3) in domain does not have an image

Example : 2

Which of the following is a function from A to B ?

- (i) $A = \{x \mid x > 0 \text{ and } x \in \mathbb{R}\}$, $B = \{y \mid y \in \mathbb{R}\}$
 (A is the set of positive real numbers and B is the set of all real numbers)
 $f = \{(x, y) \mid y = \sqrt{x}\}$
- (ii) $A = \{x \mid x \in \mathbb{R}\}$, $B = \{y \mid y \in \mathbb{R}\}$
 $f = \{(x, y) \mid y = \sqrt{x}\}$

Solution

- (i) f is a function from A to B because every element of domain (+ve reals) has a unique image (square root) in codomain
- (ii) f is not a function from A to B because –ve real nos. are present in domain and they do not have any image in codomain
 ($\because y = \sqrt{x}$ is meaningless for –ve reals of x)

Example : 3

Check the following functions for injective and surjective

- (i) $f : \mathbb{R} \rightarrow \mathbb{R}$ and $f(x) = x^2$
 (ii) $f : \mathbb{R} \rightarrow \mathbb{R}^+$ and $f(x) = x^2$
 (iii) $f : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ and $f(x) = x^2$

Solution

- (i) Injective
 Let $f(x_1) = f(x_2) \Rightarrow x_1^2 = x_2^2 \Rightarrow x_1 = \pm x_2$
 \Rightarrow it is not necessary that $x_1 = x_2$
 \Rightarrow It is not injective
- Surjective
 $y = x^2$
 $\Rightarrow x = \pm \sqrt{y}$ for –ve values of y in codomain, there does not exist any value of x in domain
 \Rightarrow It is not surjective
- (ii) Injective
 Let $f(x_1) = f(x_2) \Rightarrow x_1^2 = x_2^2 \Rightarrow x_1 = \pm x_2 \Rightarrow$ not injective
- Surjective
 $y = x^2 \Rightarrow x = \pm \sqrt{y}$
 As the codomain contains only positive real numbers, there exists some x for every values of y
 \Rightarrow it is surjective
- (iii) Injective
 $f(x_1) = f(x_2) \Rightarrow x_1^2 = x_2^2 \Rightarrow x_1 = x_2$ because domain contains only +ve reals
 \Rightarrow it is injective
- Surjective
 $y = x^2 \Rightarrow x = \pm \sqrt{y}$
 for +ve values of y , there exists some x , As codomain is \mathbb{R}^+ , it is surjective

Example : 4

Let $A = \mathbb{R} - \{3\}$ and $B = \mathbb{R} - \{1\}$

Let $f : A \rightarrow B$ be defined by $f(x) = \frac{x-2}{x-3}$

Is f bijective ?

Solution

Injective

Let $f(x_1) = f(x_2)$ where $x_1, x_2 \in A$

$$\Rightarrow \frac{x_1-2}{x_1-3} = \frac{x_2-2}{x_2-3} \quad \Rightarrow \quad (x_1-2)(x_2-3) = (x_2-2)(x_1-3) \quad (\text{because } x_1, x_2 \neq 3)$$

$$\Rightarrow x_1 = x_2 \quad (\text{on simplification})$$

Hence $f(x)$ is injective

Surjective

$$y = \frac{x-2}{x-3}$$

$$\Rightarrow y(x-3) = x-2$$

$$\Rightarrow x = \frac{3y-2}{y-1}$$

For $y \neq 1$, there exists some value of x . As the codomain does not contain 1, we have some value of x in domain for every value of y in codomain

\Rightarrow it is surjective

Hence $f(x)$ is bijective

Inverse of $f(x)$

$$\text{Interchanging } x \text{ and } y \text{ in } y = f(x) \text{ we have } x = \frac{y-2}{y-3} \quad \Rightarrow \quad y = \frac{3x-2}{x-1}$$

$$\Rightarrow f^{-1}(x) = \frac{3x-2}{x-1} \text{ is the inverse of } f(x)$$

Example : 5

Is $f : \mathbb{R} \rightarrow \mathbb{R}$, $f(x) = \cos(5x+2)$ invertible ?

Solution

Injective

Let $f(x_1) = f(x_2)$ where $x_1, x_2 \in \mathbb{R}$

$$\Rightarrow \cos(5x_1+2) = \cos(5x_2+2)$$

$$\Rightarrow 5x_1+2 = 2n\pi \pm (5x_2+2)$$

$$\Rightarrow \text{it is not necessary that } x_1 = x_2$$

hence if it not injective

Surjective

$$y = \cos(5x+2)$$

$$\Rightarrow x = \frac{\cos^{-1}y - 2}{5}$$

$$\Rightarrow \text{there is no value of } x \text{ for } y \in (-\infty, -1) \cup (1, +\infty)$$

As this interval is included in codomain, there are some values of y in codomain for which there does no exist any value of x . Hence it is not surjective.

As f is neither injective nor surjective, it is not invertible.

Example : 6

- (i) Let $f(x) = x - 1$ and $g(x) = x^2 + 1$.
What is $f \circ g$ and $g \circ f$?
- (ii) $f = \{(1, 2), (3, 5), (4, 1)\}$ and $g = \{(2, 3), (5, 1), (1, 3)\}$
write down the pairs in the mappings $f \circ g$.

Solution

- (i) $f \circ g = f[g(x)] = f(x^2 + 1) = x^2 + 1 - 1 = x^2$
 $g \circ f = g[f(x)] = g[x - 1] = (x - 1)^2 + 1$
- (ii) domain of $f \circ g$ is the domain of $g(x)$ i.e. $\{2, 5, 1\}$
 $f \circ g(2) = f[g(2)] = f(3) = 5$
 $f \circ g(5) = f[g(5)] = f(1) = 2$
 $f \circ g(1) = f[g(1)] = f(3) = 5$
 $\Rightarrow f \circ g = \{(2, 5), (5, 2), (1, 5)\}$

Example : 7

If $A = \left\{x : \frac{\pi}{6} \leq x \leq \frac{\pi}{3}\right\}$ and $f(x) = \cos x - x(1 + x)$. Find $f(A)$.

Solution

We have to find the range with A as domain.
As $f(x)$ is decreasing in the given domain

$$\frac{\pi}{6} \leq x \leq \frac{\pi}{3} \quad \Rightarrow \quad f\left(\frac{\pi}{6}\right) \geq f(x) \geq f\left(\frac{\pi}{3}\right)$$

$$\Rightarrow f(x) \in \left[\frac{1}{2} - \frac{\pi}{3} - \frac{\pi^2}{9}, \frac{\sqrt{3}}{2} - \frac{\pi^2}{36} \right]$$

$$\Rightarrow \text{the range is the interval : } \left[\frac{1}{2} - \frac{\pi}{3} - \frac{\pi^2}{9}, \frac{\sqrt{3}}{6} - \frac{\pi}{6} - \frac{\pi^2}{36} \right]$$