Example : 1

Evaluate the following integrals

Hint : Express Integrals in terms of standard results :

(1) $\int \sec^2(2-3x)\ dx = \frac{-1}{3}\ \tan(2-3x) + C$

(2) $\int \frac{\sin(2-3x)}{\cos^2(2-3x)}\ dx = \int \sec(2-3x)\tan(2-3x)\ dx = \frac{1}{-3}\ \sec(2-3x) + C$

(3) $\int e^{2x-3}dx = \frac{1}{2}\ e^{2x-3} + C$

(4) $\int \sec(2-3x)\ dx = \frac{1}{-3}\ \log|\sec(2-3x) + \tan(2-3x)| + C$

(5) $\int \frac{1}{\sqrt{4x+1}}\ dx = \frac{1}{4}\ \left(2\sqrt{4x+1}\right) + C$

(6) $\int \frac{dx}{(1-2x)^3} = \frac{1}{-2}\ \left(\frac{-1}{2(1-2x)^2}\right) + C$

Example : 2

Evaluate the following integrals

Hint : Express numerator in terms of denominator

(1) $\int \frac{x-1}{x+1}\ dx = \int \frac{x+1-2}{x+1}\ dx = \int \left(1-\frac{2}{x+1}\right) dx = \int dx - 2\int \frac{dx}{x+1} = x - 2\log|x+1| + C$

(2) $\int \frac{x^2-1}{x^2+1}\ dx = \int \frac{x^2+1}{x^2+1}\ dx - \int \frac{2dx}{x^2+1} = x - 2\tan^{-1}x + C$

(3) $\int \frac{x}{(2x+1)^2}\ dx = \frac{1}{2}\int \frac{2x+1-1}{(2x+1)^2}\ dx = \frac{1}{2}\int \frac{dx}{(2x+1)} - \frac{1}{2}\int \frac{dx}{(2x+1)^2}$

$$= \frac{1}{2}\left(\frac{1}{2}\log|2x+1|\right) - \frac{1}{2}\left[\frac{1}{2}\left(\frac{-1}{2x+1}\right)\right] + C$$

(4) $\int \frac{x^4+1}{x^2+1}\ dx = \int \frac{x^4-1}{x^2+1}\ dx + \int \frac{2}{x^2+1}\ dx\ \int (x^2-1)\ dx + 2\int \frac{dx}{x^2+1} = x^3/3 - x + 2\tan^{-1}x + C$

(5) $\int \frac{x^7}{x+1}\ dx = \int \frac{x^7+1}{x+1}\ dx - \int \frac{dx}{x+1} = \int \frac{(x+1)(x^6-x^5+x^4-x^3+x^2-x+1)}{x+1}\ dx - \log|x+1|$

$$= \frac{x^7}{7} - \frac{x^6}{6} + \frac{x^5}{5} - \frac{x^4}{4} + \frac{x^3}{3} - \frac{x^2}{2} + x - \log|x+1| + C$$

(6) $\int \frac{x^3}{(x+1)^2}\,dx = \int \frac{x^3+1}{(x+1)^2}\,dx - \int \frac{dx}{(x+1)^2} = \int \frac{x^2-x+1}{(x+1)}\,dx - \int \frac{dx}{(x+1)^2}$

$= \int \frac{x^2+x}{x+1}\,dx + \int \frac{1-2x}{x+1}\,dx - \int \frac{dx}{(x+1)^2} = |x\,dx - 2| \int \frac{x+1}{x+1}\,dx + \int \frac{3dx}{x+1} - \int \frac{dx}{(x+1)^2}$

$= \frac{x^2}{2} - 2x + 3\log|x+1| + \frac{1}{x+1} + C$

(7) $\int \frac{ax+b}{cx+d}\,dx = \frac{a}{c} \int \frac{(cx+d) - \left(d - \frac{bc}{a}\right)}{cd+d}\,dx$

$= \frac{a}{c} \int dx - \frac{a}{c} \frac{\left(d - \frac{bc}{a}\right)}{cx+d}\,dx = \frac{ax}{c} - \left(\frac{ad-bc}{c^2}\right) \log|cx\ + d| + C$

Example : 3

Evaluate the following integrals

Hint : Use $\int [f(x)]^n\ f'(x)\,dx = \frac{[f(x)]^{n+1}}{n+1} + C$

(1) $\int \frac{(\sin^{-1}x)^3}{\sqrt{1-x^2}}\,dx = \frac{1}{4}(\sin^{-1}x)^4 + C$

(2) $\int \sec^4 x \tan x\,dx = \int \sec^3 x(\sec x\,\tan x)\,dx = \frac{\sec^4 x}{4} + C$

(3) $\int \frac{\log^n x}{x}\,dx = \frac{\log^{n+1} x}{n+1} + C$

(4) $\int \frac{x}{(x^2+1)^3}\,dx = \frac{1}{2} \int \frac{1}{(x^2+1)^3}\,2x\,dx = \frac{1}{2}\,\frac{(x^2+1)^{-2}}{-2} + C$

(5) $\int \sin^5 x \cos x\,dx = \frac{\sin^6 x}{6} + C$

Example : 4

Evaluate the following integrals

Hint : Use $\int \frac{f'(x)}{f(x)}\,dx = \log|f(x)| + C$

(1) $\int \frac{x^3}{1+x^4}\,dx = \frac{1}{4} \int \frac{4x^3}{1+x^4}\,dx = \frac{1}{4}\log|1+x^4| + C$

(2) $\int \frac{e^x - e^{-x}}{e^x + e^{-x}}\, dx = \log |e^x + e^{-x}| + C$

(3) $\int \frac{e^x + 1}{e^x - 1}\, dx = \int \frac{e^{x/2} + e^{-x/2}}{e^{x/2} - e^{-x/2}} = 2 \int \frac{\frac{1}{2}e^{x/2} + \frac{1}{2}e^{-x/2}}{e^{x/2} - e^{-x/2}} = 2 \log |e^{x/2} - e^{-x/2}| + C$

(4) $\int \frac{x^3}{(x^2+1)}\, dx = \int \frac{x^3 + x}{(x^2+1)^2}\, dx - \int \frac{x\, dx}{(x^2+1)^2} = \int \frac{x}{x^2+1}\, dx - \int \frac{x}{(x^2+1)^2}\, dx$

$= \frac{1}{2} \int \frac{2x}{x^2 - 1}\, dx - \frac{1}{2} \int \frac{2x}{(x^2+1)^2}\, dx = \frac{1}{2} \log |x^2 + 1| - \frac{1}{2} \left(\frac{-1}{x^2+1} \right) + C$

(5) $\int \frac{dx}{a + be^x} = \int \frac{e^{-x}}{ae^{-x} + b}\, dx - \frac{1}{a} \int \frac{-ae^{-x}}{ae^{-x} + b}\, dx = -\frac{1}{a} \log |ae^{-x} + b| + C$

(6) $\int \frac{\tan x + 1}{\tan x - 1}\, dx = \int \frac{\sin x + \cos x}{\sin x - \cos x}\, dx = \log |\sin x - \cos x| + C$

(7) $\int \frac{\sec x}{\log(\sec x + \tan x)}\, dx = \log | \log (\sec x + \tan x) | + C$

Note that $\frac{d}{dx} \log (\sec x + \tan x) = \sec x$

(8) $\int \frac{x^2 - 1}{x(x^2+1)}\, dx \int \frac{1 - \frac{1}{x^2}}{x + \frac{1}{x}}\, dx = \log \left| x + \frac{1}{x} \right| + C$

(9) $\int \frac{dx}{x + \sqrt{x}} = \int \frac{dx}{\sqrt{x}\left(\sqrt{x} + 1\right)} = 2 \int \frac{\frac{1}{2\sqrt{x}}}{\left(\sqrt{x} + 1\right)}\, dx = 2 \log | \sqrt{x} + 1| + C$

(10) $\int \frac{x}{(x^4+1)\tan^{-1} x^2}\, dx = \frac{1}{2} \int \frac{\frac{2x}{1+x^4}}{\tan^{-1} x}\, dx = \frac{1}{2} \log |\tan^{-1} x^2| + C$

(11) $\int \frac{dx}{x \log x \log \log x} = \int \frac{\frac{1}{x \log x}}{\log \log x}\, dx = \log |\log \log x| + C$

(12) $\int \frac{\sin 2x}{1 + \sin^2 x}\, dx = \log |1 + \sin^2 x| + C$

(13) $\int \frac{e^{x-1} + x^{e-1}}{e^x + x^e}\, dx = \frac{1}{e} \int \frac{e^x + ex^{e-1}}{e^x + x^e}\, dx = \frac{1}{e} \log |e^x + x^e| + C$

Example : 5

Evaluate

(1) $\int \sin^2 x\, dx$ **(2)** $\int \sin^3 x\, dx$ **(3)** $\int \sin^4 x\, dx$ **(4)** $\int \sin^4 x \cos^4 x\, dx$

Hint : Reduce the degree of integral and to one by transforming it into multiple angles of sine and cosine.

Solution

(1) $\int \sin^2 x\, dx = \int \frac{1-\cos 2x}{2}\, dx = \frac{1}{2}\left[x - \frac{\sin 2x}{2}\right] + C$

(2) $\int \sin^3 x\, dx = \int \frac{3\sin x - \sin 3x}{4}\, dx = \frac{1}{4}\left[\int 3\sin x\, dx - \int \sin 2x\, dx\right] dx = \frac{1}{4}\left[-3\cos x + \frac{\cos 3x}{3}\right] + C$

$= \frac{-3}{4}\cos x + \frac{1}{12}\cos 3x + C$

(3) $\int \sin^4 x\, dx = \int \left(\frac{1-\cos 2x}{2}\right)^2 dx = \frac{1}{4}\int (1-2\cos 2x)\, dx + \frac{1}{4}\int (\cos^2 2x\, dx)$

$= \frac{x}{4} - \frac{1}{4}\sin 2x + \frac{1}{8}\int (1+\cos 4x)\, dx$

$= \frac{x}{4} - \frac{1}{4}\sin 2x + \frac{x}{8} + \frac{\sin 4x}{32} + C$

(4) $\int \sin^4 x \cos^4 x\, dx = \frac{1}{16}\int \sin^4 2x$. Now proceed on the pattern of $\int \sin^4 x\, dx$

Example : 6

Evaluate :

(1) $\int \frac{dx}{1+\sin x}$ **(2)** $\int \frac{dx}{1+\cos x}$

Solution

(1) $\int \frac{dx}{1+\sin x} = \int \frac{1-\sin x}{\cos^2 x}\, dx = \int \sec^2 x\, dx - \int \sec x \tan x\, dx = \tan x - \sec x + C$

Alternative Method

$\int \frac{dx}{1+\sin x} = \int \frac{dx}{1+\cos\left(\frac{\pi}{2}-x\right)} = \int \frac{dx}{2\cos^2\left(\frac{\pi}{4}-\frac{x}{2}\right)} = \frac{1}{2}\int \sec^2\left(\frac{\pi}{4}-\frac{\pi}{2}\right) dx = \frac{1}{2}\ \frac{\tan\left(\frac{\pi}{4}-\frac{x}{2}\right)}{-1/2} + C$

$= -\tan\left(\frac{x}{4}-\frac{x}{2}\right) + C$

(2) $\int \frac{dx}{1+\cos x} = \int \frac{1-\cos x}{\sin^2 x} = \int \text{cosec}^2 x\, dx - \int \text{cosec}\, x \cot x\, dx = -\cot x + \text{cosec}\, x + C$

Alternative Method

$\int \frac{dx}{1+\cos x} = \int \frac{dx}{2\cos^2 x/2} = \frac{1}{2}\int \sec^2 \frac{x}{2}\, dx = \frac{1}{2}\ \frac{\tan x/2}{1/2} + C = \tan\frac{x}{2} + C$

Example : 7

Evaluate :

(1) $\int \sin 2x \sin 3x \, dx$ **(2)** $\int \sin 2x \sin 4x \cos 5x \, dx$ **(3)** $\int \sin^2 x \cos^2 x \, dx$

Hint : Apply trigonometric formulas to convert product form of the integrand into sum of sines and cosines of multiple angle

Solution

(1) $\int \sin 2x \sin 3x \, dx = \frac{1}{2}\int 2\sin 2x \sin 3x \, dx = \frac{1}{2}\int (\cos x - \cos 5x) \, dx = \frac{1}{2} \sin x - \frac{1}{10} \sin 5x + C$

(2) $\int \sin 2x \cos 4x \cos 5x \, dx = \frac{1}{2}\int (2\sin 2x \cos 4x) \cos 5x \, dx$

$= \frac{1}{2}\int (\sin 6x - \sin 2x) \cos 5x \, dx = \frac{1}{4}\int 2\sin 6x \cos 5x \, dx - \frac{1}{4}\int 2\sin 2x \cos 5x \, dx$

$= \frac{1}{4}\int (\sin 11x + \sin x) \, dx - \frac{1}{4}\int (\sin 7x - \sin 3x) \, dx$

$= -\frac{1}{4}\,\frac{\cos 11x}{11} - \frac{1}{4}\cos x + \frac{1}{4}\,\frac{\cos 7x}{7} - \frac{1}{4}\,\frac{\cos 3x}{3}$

(3) $\int \sin^2 x \cos^2 x \, dx = \frac{1}{4}\int \sin^2 2x \, dx = \frac{1}{4}\int \frac{1-\cos 4x}{2} \, dx = \frac{1}{8}\left(x - \frac{\sin 4x}{4}\right) + C$

Example : 8

Evaluate : $\int \frac{dx}{a\sin x + b\cos x}$

Solution

$$\int \frac{dx}{a\sin x + b\cos x} = \frac{1}{\sqrt{a^2+b^2}} \int \frac{dx}{\frac{a}{\sqrt{a^2+b^2}}\sin x + \frac{b}{\sqrt{a^2+b^2}}\cos x}$$

$$= \frac{1}{\sqrt{a^2+b^2}} \int \frac{dx}{\sin x \cos\alpha + \cos x \sin\alpha} \qquad \text{where } \alpha = \tan^{-1}(b/x)$$

$$= \frac{1}{\sqrt{a^2+b^2}} \int \operatorname{cosec}(x+\alpha) \, dx$$

$$= \frac{1}{\sqrt{a^2+b^2}} \log|\operatorname{cosec}(x+\alpha) - \cot(x+\alpha)| + C \qquad \text{where } \alpha = \tan^{-1}(b/a)$$

Example : 9

Evaluate $\int e^x (x+1) \cos(xe^x) \, dx$

Solution

The given integral is in terms of the variable x, we can simplify the integral by connecting it in the terms of another variable t using substitution

Here let us put $x e^x = t$

and hence $xe^x \, dx + e^x \, dx = dt$

$\Rightarrow$ $e^x (x + 1) \, dx = dt$

The given integral = $\int \cos(xe^x)[e^x(x+1)\,dx] = \int \cos t\,dt$ = sin t + C = sin (x e^x) + C

Note that the final result of the problem must be in terms of x.

Example : 10

Evaluate :

(1) $\int \frac{x^2}{1+x^6}\,dx$ **(2)** $\frac{dx}{x+\sqrt[3]{x}}$ **(3)** $\int \frac{x^2}{\sqrt{1+x}}\,dx$

Solution

(1) Let $x^3 = t$ $\Rightarrow$ $3x^2\,dx = dt$

$$\Rightarrow \int \frac{x^2dx}{1+x^6} = \frac{1}{3}\int \frac{3x^2dx}{1+x^6} = \frac{1}{3}\int \frac{dt}{1+t^2} = \frac{1}{3}\tan^{-1} t + C = \frac{1}{3}\tan^{-1} x^3 + C$$

(2) $\sqrt[3]{x}$ indicates that we should try $x = t^3$

$\Rightarrow$ $dx = 3t^2\,dt$

$$\Rightarrow \int \frac{dx}{x+\sqrt[3]{x}} = \int \frac{3t^2dt}{t^3+t} = 3\int \frac{t\,dt}{t^2+1} = \frac{3}{2}\int \frac{2t\,dt}{t^2+1} = \frac{3}{2}\log|t^2+1| + C = \frac{3}{2}\log|x^{2/3}+1| + C$$

(3) Let $1 + x = t^2$ $\Rightarrow$ $dx = 2t\,dt$

$$\Rightarrow \int \frac{x^2}{\sqrt{1+x}}\,dx = \int \frac{(t^2-1)^2}{\sqrt{t^2}}\,2t\,dt = \int \frac{t^4+1-2t^2}{t}\,2t\,dt$$

$$= 2\frac{t^5}{5} + 2t - \frac{4t^3}{3} + C = \frac{2}{5}(1+x)^{5/2} + 2\sqrt{1+x} - \frac{4}{3}(1+x)^{3/2} + C$$

Example : 11

(1) $\int \frac{a^{x+\tan^{-1}a^x}}{a^{2x}+1}\,dx$ **(2)** $\int \sqrt{\sin\theta}\cos^3\theta\,d\theta$ **(3)** $\int \sqrt{\frac{x}{1-x^3}}\,dx$

Solution

(1) The given integral can be written as : $\int \frac{a^{x+\tan^{-1}a^x}}{a^{2x}+1}\,dx$

Let $\tan^{-1} a^x = t$

$$\Rightarrow \frac{1}{1+a^{2x}}\,a^x \log a\,dx = dt$$

$$\Rightarrow I = \int \frac{a^{\tan^{-1}a^x}\,a^x \log a\,dx}{\log a\,(1+a^{2x})} = \int \frac{a^t\,dt}{\log a}$$

$$\Rightarrow I = \frac{1}{\log a}\,\frac{a^t}{\log a} + C$$

$$\Rightarrow I = \frac{a^{\tan^{-1}a^x}}{(\log a)^2} + C$$

(2) Let $\sin\theta = t^2$ $\Rightarrow$ $\cos\theta\,d\theta = 2t\,dt$

$$\Rightarrow \int \sqrt{\sin\theta}\cos^3\theta\,d\theta = \int \sqrt{\sin\theta}\,(1-\sin^2\theta)\cos\theta\,d\theta$$

$= \int \sqrt{t^2}\ (1 - t^4)\ 2t\ dt = 2 \int (t^2 - t^6)\ dt$

$= \dfrac{2t^3}{3} - \dfrac{2t^7}{7} + C = \dfrac{2}{3}\ (\sin\theta)^{3/2} - \dfrac{2}{7}\ (\sin q)^{7/2} + C$

(3) The given integral is $I = \int \dfrac{\sqrt{x}\ dx}{\sqrt{1 - x^3}}$

$\sqrt{x}$ appears in the derivative of $x^{3/2}$

hence, let $x^{3/2} = t \quad \Rightarrow \quad 3/2\ \sqrt{x}\ dx = dt$

$\Rightarrow \quad I = \dfrac{2}{3} \int \dfrac{\frac{3}{2}\sqrt{x}\ dx}{\sqrt{1 - x^3}} = \dfrac{2}{3} \int \dfrac{dt}{\sqrt{1 - t^2}} = \dfrac{2}{3}\ \sin^{-1} t + C = \dfrac{2}{3}\ \sin^{-1} x^{3/2} + C$

Example : 12

Evaluate the following integrals

(1) $\int \tan^2 x\ dx = \int (\sec^2 x - 1)\ dx = \tan x - x + C$

(2) $\int \tan^3 x\ dx = \int \tan x\ (\sec^2 x - 1)\ dx = \int \tan x \sec^2 x\ dx - \int \tan x\ dx = \dfrac{\tan^2 x}{2} - \log|\sec x| + C$

(3) $\int \tan^4 x\ dx = \int \tan^2 x\ (\sec^2 x - 1)\ dx = \int \tan^2 x\ (\sec^2 x\ dx) - \int \tan^2 x\ dx$

$= \dfrac{\tan^3 x}{3} - \int \sec^2 x\ dx + \int dx = \dfrac{\tan^3 x}{3} - \tan x + x + C$

(4) $\int \sec^4 x\ dx = \int \sec^2 x \sec^2 x\ dx = \int (1 + \tan^2 x)\ \sec^2 x\ dx$

$= \int (1 + t^2)\ dt = t + t^3/3 + C = \tan x = \dfrac{\tan^3 x}{3} + C$

Example : 13

Evaluate :

(1) $\int \sin^3 x \cos^4 x\ dx$ **(2)** $\int \sin^5 x\ dx$

Solution

(1) $\int \sin^3 x \cos^4 x\ dx = \int \sin^2 x \cos^4 x\ (\sin x\ dx)$

$= -\int (1 - t^2)\ t^4\ dt \quad$ where $t = \cos x$

$= \dfrac{t^7}{7} - \dfrac{t^5}{5} + C = \dfrac{\cos 7x}{7} - \dfrac{\cos^5 x}{5} + C$

(2) $\int \sin^5 x\, dx = \sin^4 x \sin x\, dx = -\int (1-\cos^2 x)^2 \;(-\sin x\, dx)$

$= -\int (1-t^2)^2\, dt$ where t = cos x

$= -\int (1+t^4-2t^2)\, dt\; - t - \frac{t^5}{5} + \frac{2t^3}{3} + C$

$= -\cos x - \frac{\cos^5 x}{5} + \frac{2}{3}\cos^3 x + C$

Example : 14

Type : $\int \frac{dx}{ax^2+bx+c}$

(1) $\int \frac{dx}{x^2+x+1} = \int \frac{dx}{x^2+2\frac{1}{2}x+\frac{3}{4}+\frac{1}{4}} = \int \frac{dx}{\left(x+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}$

$= \frac{1}{\sqrt{3}/2}\tan^{-1}\left(\frac{x+1/2}{\sqrt{3}/2}\right) + = \frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{2x+1}{\sqrt{3}}\right) + C$

(2) $\int \frac{dx}{1-4x-2x^2} = \frac{1}{2}\int \frac{dx}{1/2-(x^2+2x)} = \frac{1}{2}\int \frac{dx}{\left(\sqrt{3/2}\right)^2-(x+1)^2}$

$= \frac{1}{2}\,\frac{1}{2\sqrt{3/2}}\log\left|\frac{\sqrt{3/2}+x+1}{\sqrt{3/2}-(x+1)}\right| + C$

$= \frac{1}{2\sqrt{6}}\log\left|\frac{\sqrt{3}+\sqrt{2}x+\sqrt{2}}{\sqrt{3}-\sqrt{2}x-\sqrt{2}}\right| + C$

(3) $\int \frac{dx}{x^2+6x+1} = \int \frac{dx}{x^2+6x+9-8} = \int \frac{dx}{(x+3)^2-\left(2\sqrt{2}\right)^2} = \frac{1}{2\left(2\sqrt{2}\right)}\log\left|\frac{x+3-2\sqrt{2}}{x+3+2\sqrt{2}}\right| + C$

Example : 15

Type : $\int \frac{dx}{\sqrt{ax^2+bx+c}}$

(1) Let $I = \int \frac{dx}{\sqrt{ax^2+bx+c}}$

Treat $1 - x - x^2$ as $1 - (x + x^2) = 1 - (x^2 + x + 1/4) + 1/4 = 5/4 - (x + 1/2)^2$

$\Rightarrow \quad I = \int \frac{dx}{\sqrt{\frac{5}{4}-\left(x+\frac{1}{2}\right)^2}} = \sin^{-1}\left(\frac{x+1/2}{\sqrt{5}/2}\right) = \sin^{-1}\left(\frac{2x+1}{\sqrt{5}}\right) + C$

Let $I = \int \frac{dx}{\sqrt{2x^2+6x+2}}$

Now $2x^2 + 6x + 2 = 2(x^2 + 3x + 1) = 2\left(x^2 + \frac{6x}{2} + \frac{9}{4} - \frac{9}{4} + 1\right) = 2\left[\left(x+\frac{3}{2}\right)^2 - \frac{5}{4}\right]$

This is in the form $x^2 - a^2$.

$$\Rightarrow \qquad I = \frac{1}{\sqrt{2}} \int \frac{dx}{\sqrt{(x+3/2)^2 - 5/4}} = \frac{1}{\sqrt{2}} \log \left| x + \frac{3}{2} + \sqrt{(x+3/2)^2 - 5/4} \right| + C$$

Example : 16

Type : $\int \sqrt{ax^2+bx+c}\ dx$

$$\text{Let } I = \int \sqrt{ax^2+5x+1}\ dx = \int \sqrt{(x+5/2)^2 - 21/4}\ dx$$

$$= \frac{x+5/2}{2}\sqrt{x^2+5x+1} - \frac{21}{8}\log\left|x+5/2+\sqrt{(x+5/2)^2-21/4}\right| + C$$

$$= \frac{2x+5}{4}\sqrt{x^2+5x+1} - \frac{21}{8}\log\left|x+5/2+\sqrt{x^2+5x+1}\right| + C$$

Example : 17

Evaluate $\int \frac{3x+1}{\sqrt{x^2+4x+1}}\ dx$

Solution

The linear expression in the numerator can be expressed as $3x + 1 = \ell\ d/dx\ (x^2 + 4x + 1) + m$

$\Rightarrow \qquad 3x + 1 = \ell\ (2x + 4) + m$

comparing the coefficients of x and x^0,

$3 = 2\ell \qquad$ and $\qquad 1 = 4\ell + m$

$\Rightarrow \qquad \ell = 3/2 \qquad$ and $\qquad m = -5$

$$\Rightarrow \qquad I = \int \frac{3x+1}{\sqrt{x^2+4x+1}} = \int \frac{3/2(2x+4)-5}{\sqrt{x^2+4x+1}}\ dx = \frac{3}{2}\int \frac{2x+4}{\sqrt{x^2+4x+1}} - 5\int \frac{dx}{\sqrt{x^2+4x+1}}$$

Let $\qquad I_1 = \frac{3}{2}\int \frac{2x+4}{\sqrt{x^2+4x+1}} = \frac{3}{2}\int \frac{dt}{\sqrt{t}} \qquad$ (where $t = x^2 + 4x + 1$)

$$= 3\sqrt{t} + C = 3\sqrt{x^2+4x+1} + C$$

Let $\qquad I_2 = 5\frac{dx}{\sqrt{x^2+4x+1}} = 5\int \frac{dx}{\sqrt{(x+2)^2-3}} = 5\log\left|x+2+\sqrt{(x+2)^2-3}\right| + C$

$$\Rightarrow \qquad I = I_1 - I_2 = 3\sqrt{x^2+4x+1} - 5\log\left|x+2+\sqrt{x^2+4x+1}\right| + C$$

Example : 18

Evaluate : $\int \frac{x^2 - x + 1}{2x^2 + x + 2}\, dx$

Solution

Express numerator in terms of denominator and its derivative

Let $x^2 - x + 1 = \ell\,(2x^2 + x + 2) + m\,(4x + 1) + n$

$\Rightarrow \quad 1 = 2\ell - 1 = \ell + 4m \qquad 1 = 2\ell + m + n$

$$\Rightarrow \quad 1 = \int \frac{x^2 - x + 1}{2x^2 + x + 2}\, dx = \int \frac{1/2(2x^2 + x + 2) - 3/8(4x + 1) + 3/8}{2x^2 + x + 2}\, dx$$

$$= \frac{1}{2} \int dx - \frac{3}{8} \int \frac{4x + 1}{2x^2 + x + 2}\, dx + \frac{3}{8} \int \frac{dx}{2x^2 + x + 2}$$

$$= \frac{x}{2} - \frac{3}{8} \log |2x^2 + x + 2| + \frac{3}{8}\, I_1 \qquad \text{where } I_1 = \int \frac{dx}{2x^2 + x + 2}$$

$$= \frac{1}{2} \int \frac{dx}{x^2 + 1/2x + 1/16 - 1/16 + 1} + \frac{1}{2} \int \frac{dx}{(x + 1/4)^2 + 15/16}$$

$$= \frac{1}{2}\, \frac{1}{\sqrt{15/4}} \tan^{-1}\left(\frac{x + 1/4}{\sqrt{15/4}}\right) + C = \frac{2}{\sqrt{15}} \tan^{-1}\left(\frac{4x + 1}{\sqrt{15}}\right) + C$$

$$= \frac{x}{2} - \frac{3}{8} \log |2x^2 + x + 2| + \frac{3}{4\sqrt{15}} \tan^{-1}\left(\frac{4x + 1}{\sqrt{15}}\right) + C$$

Example : 20

$$\int \frac{dx}{3\sin^2 x + 4\cos^2 x}$$

Solution

$$\int \frac{dx}{3\sin^2 x + 4\cos^2 x} = \int \frac{\sec^2 x}{3\tan^2 x + 4}\, dx = \int \frac{dt}{3t^2 + 4} \qquad \text{where } t = \tan x$$

$$= \frac{1}{3} \int \frac{dt}{t^2 + (2/\sqrt{3})^2} = \frac{1}{3}\, \frac{1}{2/\sqrt{3}} \tan^{-1}\left(\frac{t}{2/\sqrt{3}}\right) + C$$

$$= \frac{2}{2\sqrt{3}} \tan^{-1}\left(\frac{\sqrt{3}}{2} \tan x\right) + C$$

Example : 21

Evaluate :

(1) $\int \frac{dx}{4 + 5\sin x}$ **(2)** $\int \frac{dx}{a + b\cos x}$ where a, b > 0

Solution

(1) $I = \int \frac{dx}{4 + 5\sin x}$

Put $\tan \frac{x}{2} = t \quad \Rightarrow \quad x = 2\tan^{-1} t$

$\Rightarrow \quad \cos x = \frac{1 - t^2}{1 + t^2}$; $\sin x = \frac{2t}{1 + t^2}$; dx $\frac{2dt}{1 + t^2}$

$$\Rightarrow \quad I = \int \frac{\frac{2dt}{1+t^2}}{4+5\left(\frac{2t}{1+t^2}\right)} = \int \frac{2dt}{4t^2+10t+4} = \frac{1}{2}\int \frac{dt}{t^2+5/2t+1}$$

$$= \frac{1}{2}\int \frac{dt}{(t+5/4)^2-9/16} = \frac{1}{2}\ \frac{1}{2\times 3/4}\ \log\left|\frac{t+5/4-3/4}{t+5/4+3/4}\right| + C$$

$$= \frac{1}{3}\log\left|\frac{4t+2}{4t+8}\right| + C = \frac{1}{3}\log\left|\frac{2\tan\frac{x}{2}+1}{2\tan\frac{x}{2}+4}\right| + C$$

(2) $\int \frac{dx}{a+b\cos x}$ where a, b > 0

Let tan x/2 = t

$$\Rightarrow \quad I = \int \frac{dx}{a+b\cos x} = \int \frac{\frac{2dt}{1+t^2}}{a+b\left(\frac{1-t^2}{1+t^2}\right)} \quad \Rightarrow \quad \int \frac{2dt}{(a+b)+(a-b)\,t^2}$$

Case – 1 Let a = b

$$\Rightarrow \quad I = \int \frac{2dt}{a+b} = \frac{2t}{a+b} = \frac{2}{a+b}\tan\frac{x}{2} + C$$

Case – 2 Let a > b

$$\Rightarrow \quad I = \frac{1}{a-b}\int \frac{2dt}{\frac{a+b}{a-b}+t^2} = \frac{2}{a-b}\sqrt{\frac{a-b}{a+b}}\ \tan^{-1}\left(t\sqrt{\frac{a-b}{a+b}}\right) + C$$

$$= \frac{2}{\sqrt{a^2-b^2}}\tan^{-1}\left(\tan\frac{x}{2}\sqrt{\frac{a-b}{a+b}}\right) + C$$

Case – 3 Let a < b

$$\Rightarrow \quad I = \int \frac{2dt}{(a+b)-(b-a)\,t^2} = \frac{2}{b-a}\int \frac{dt}{\frac{b+a}{b-a}-t^2}$$

$$= \frac{2}{b-a}\ \frac{\sqrt{b-a}}{\sqrt{b+a}}\ \log\left|\frac{\sqrt{\frac{b+a}{b-a}}+t}{\sqrt{\frac{b+a}{b-a}}-t}\right| + C$$

$$= \frac{1}{\sqrt{b^2-a^2}}\log\left|\frac{\sqrt{b+a}+\sqrt{b-a}\tan\frac{x}{2}}{\sqrt{b+a}-\sqrt{b-a}\tan\frac{x}{2}}\right| + C$$

Example : 22

Evaluate : $\int \frac{2\sin x + 3\cos x}{\sin x + 4\cos x}$ dx

Solution

Express numerator as the sum of denominator and its derivative

Let 2 sin x + 3 cos x = ℓ (sin x + 4 cos x) + m (cos x – 4 sin x)

comparing coefficients of sin x and cos x

$2 = \ell - 4m, \qquad 3 = 4\ell + m$

$\Rightarrow \quad \ell = 14/17 \qquad m = -5/17$

$$\Rightarrow \quad I = \int \frac{2\sin x + 3\cos x}{\sin x + 4\cos x}\, dx$$

$$\Rightarrow \quad I = \frac{14}{17} \int \frac{\sin x + 4\cos x}{\sin x + 4\cos x}\, dx - \frac{5}{17} \int \frac{\cos x - 4\sin x}{\sin x + 4\cos x}\, dx$$

$$\Rightarrow \quad I = \frac{14}{17}\, x - \frac{5}{17} \log \ |\sin x + 4 \cos x) + C$$

Example : 23

Evaluate

(1) $\int \frac{x^2 + 1}{x^4 + 1}$ dx **(2)** $\int \frac{x^2 - 1}{x^4 + 1}$ dx

Solution

(1) $$\text{Let } I_1 = \int \frac{x^2+1}{x^4+1}\, dx = \int \frac{1 + \frac{1}{x^2}}{x^2 + \frac{1}{x^2}}\, dx = \int \frac{1 + \frac{1}{x^2}}{\left(x - \frac{1}{x}\right)^2 + 2}\, dx = \int \frac{dt}{t^2 + 2}$$

$$\text{where } t = x - \frac{1}{x} = \frac{1}{\sqrt{2}} \tan^{-1}\left(\frac{t}{\sqrt{2}}\right) + C = \frac{1}{\sqrt{2}} \tan^{-1}\left(\frac{x^2 - 1}{x\sqrt{2}}\right) + C$$

(2) $$\text{Let } I_2 = \int \frac{x^2-1}{x^4+1}\, dx = \int \frac{1 - \frac{1}{x^2}}{x^2 + \frac{1}{x^2}}\, dx = \int \frac{1 - \frac{1}{x^2}}{\left(x + \frac{1}{x}\right)^2 - 2}\, dx$$

$$\text{Let } I_2 = \int \frac{dt}{t^2 - 2} \qquad \text{where } t = x + \frac{1}{x} = \frac{2}{2\sqrt{2}} \log \left| \frac{t - \sqrt{2}}{t + \sqrt{2}} \right| + C$$

$$= \frac{1}{2\sqrt{2}} \log \left| \frac{x^2 - x\sqrt{2} + 1}{x^2 + x\sqrt{2} + 1} \right| + C$$

Example : 24

Evaluate : $\int \frac{x^2-1}{(x^2+1)\sqrt{x^4+1}}\,dx$

Solution

The given integral is

$$I = \int \frac{1-\frac{1}{x^2}}{\frac{x^2+1}{x}\sqrt{\frac{x^4+1}{x^2}}}\,dx = \int \frac{1-\frac{1}{x^2}}{\left(x+\frac{1}{x}\right)\sqrt{x^2+\frac{1}{x^2}}}\,dx = \int \frac{dt}{t\sqrt{t^2-2}} \qquad \text{where } x+\frac{1}{x} = t$$

$$\Rightarrow \qquad I = \frac{1}{\sqrt{2}}\sec^{-1}\left|\frac{t}{\sqrt{2}}\right| + C = \frac{1}{\sqrt{2}}\sec^{-1}\left|\frac{x^2+1}{x\sqrt{2}}\right| + C$$

Example : 25

Evaluate : $\int x\cos x\,dx$

Solution

$$I = \int \underset{\text{part 1}}{x}\ \underset{\text{part 2}}{\cos dx} = x\int \cos x\,dx - \int\left[\int \cos x\,dx\right]dx$$

$$\Rightarrow \qquad I = x\sin x - \int \sin x\,dx = x\sin x + \cos x + C$$

Example : 26

Study the following examples carefully

(1) $\int x\sec^2 x\,dx \quad = x\int \sec^2 x\,dx - \int \tan x\,dx = x\tan x - \log|\sec x| + C$

(2) $\int \sin^{-1} dx \quad = \sin^{-1}\int dx - \int x\frac{1}{\sqrt{1-x^2}}\,dx = x\sin^{-1}x - \frac{1}{2}\int \frac{2x}{\sqrt{1-x^2}}\,dx$

$= x\sin^{-1}x + \frac{1}{2}\int \frac{dt}{\sqrt{t}} \qquad \text{where } I - x^2 = t$

$= x\sin^{-1}x + \frac{1}{2}\,2\sqrt{t} + C = x\sin^{-1}x + \sqrt{1-x^2} + C$

(3) $\int \tan^{-1}x\,dx \quad = \tan^{-1}\int dx - \int \frac{x}{1+x^2}\,dx$

$= x\tan^{-1}x - \frac{1}{2}\log|1+x^2| + C$

(4) $\int x\,e^x\,dx \quad = x\int e^x\,dx - \int e^x\,dx = x\,e^x - e^x + C$

(5) $\int \log x\,dx \quad = \log x\int dx - \int x\,\frac{1}{x}\,dx = x\log x - x + C$

(6) $\int x^2 \sin x\, dx \quad = x^2 \int \sin x\, dx - \int (-\cos x)\, 2x\, dx$

$= -x^2 \cos x + 2 \int x \cos x\, dx$

$= -x^2 \cos x + 2 \left[x \int \cos x\, dx - \int \sin x\, dx \right]$

$= -x^2 \cos x + 2x \sin x + 2 \cos x + C$

Example : 27

Evaluate : $\int x \sin^{-1} dx$

Solution

$$\int x \sin^{-1} dx \quad = \sin^{-1} x \int x\, dx - \int \frac{x^2 dx}{2\sqrt{1-x^2}} = \frac{x^2}{2} \sin^{-1} x + \frac{1}{2} \int \frac{1-x^2-1}{\sqrt{1-x^2}}\, dx$$

$$= \frac{x^2}{2} \sin^{-1} x + \frac{1}{2} \int \sqrt{1-x^2}\, dx - \frac{1}{2} \int \frac{dx}{\sqrt{1-x^2}}$$

$$= \frac{x^2}{2} \sin^{-1} x + \frac{1}{2} \left[\frac{x}{2}\sqrt{1-x^2} + \frac{1}{2}\sin^{-1} x \right] - \frac{1}{2} \sin^{-1} x + C$$

Example : 28

Evaluate : $e^x \sin x\, dx$

Solution

Let $I = \int e^x \sin x\, dx$

$\Rightarrow \quad I = \int \sin x\, e^x dx = \sin x \int e^x\, dx - \int e^x\, [\cos x\, dx]$

$\Rightarrow \quad I = e^x \sin x - \int \cos x\, e^x\, dx$

$\Rightarrow \quad I = e^x \sin x - [\cos x \int e^x dx - \int e^x\, (-\sin x\, dx)]$

$\Rightarrow \quad I = e^x (\sin x - \cos x) - \int e^x \sin x\, dx$

$\Rightarrow \quad I = e^x (\sin x - \cos x) - I$

$\Rightarrow \quad I + I = e^x (\sin x - \cos x)$

$\Rightarrow \quad I = e^x/2\, (\sin x - \cos x) + C$

Example : 29

Evaluate : $\int \sec^3 x\, dx$

Solution

Let $I = \int \sec^3 x\, dx = \int \sec x \sec^2 x\, dx = \sec x \int \sec^2 x - \int \tan x\, (\sec x \tan x)\, dx$

$\Rightarrow \quad I = \sec x \tan x - \int \sec x\, (\sec^2 x - 1)\, dx = \sec x \tan x - \int \sec^3 x\, dx + \int \sec x\, dx$

$\Rightarrow \quad I = \sec x \tan x - I + \log |\sec x + \tan x|$

$\Rightarrow \quad I = 1/2\, [\sec x \tan x + \log| \sec x + \tan x| + C$

Example : 30

Evaluate : **(1)** $\int e^x\left[\frac{2+\sin 2x}{1+\cos 2x}\right] dx$ **(2)** $\int \frac{xe^x}{(1+x)^2}\ dx$

(3) $\int \frac{e^x(x^2+1)}{(x+1)^2}\ dx$ **(4)** $\int \left[\log(\log x)+\frac{1}{\log x}\right] dx$

Solution

(1) $I = \int e^x\left[\frac{2+\sin 2x}{1+\cos 2x}\right] dx$

$\Rightarrow \quad I = \int e^x\left[\frac{2}{1+\cos 2x}+\frac{\sin 2x}{1+\cos 2x}\right] dx$

$\Rightarrow \quad I = e^x\left[\frac{2}{2\cos^2 x}+\frac{2\sin x\cos x}{2\cos^2 x}\right] dx = \int e^x\ [\sec^2 + \tan x]\ dx$

$\Rightarrow \quad I = \int e^x\ [\tan x+\sec^2 x]\ dx = e^x \tan x + C$

(2) $I = \int \frac{xe^x}{(1+x)^2}\ dx = \left[\frac{1+x-1}{(1+x)^2}\right] dx$

$\Rightarrow \quad I = \int e^x\left[\frac{1}{1+x}-\frac{1}{(1+x)^2}\right] dx = e^x\left(\frac{1}{1+x}\right) + C$

(3) $I = \int \frac{e^x(x^2+1)}{(x+1)^2}\ dx = \int\left[\frac{x^2-1}{(x+1)^2}+\frac{2}{(x+1)^2}\right] dx$

$\Rightarrow \quad I = \int e^x\left[\frac{x-1}{x+1}+\frac{2}{(x+1)^2}\right] dx$

We now se that $\frac{d}{dx}\left(\frac{x-1}{x+1}\right) = \frac{(x+1)-(x-1)}{(x+1)^2} = \frac{2}{(x+1)^2}$

$\Rightarrow \quad I = e^x\left[\frac{x-1}{x+1}\right] + C$

(4) $I = \int\left[\log(\log x)+\frac{1}{\log x}\right] dx$

Substitute log x = 1 $\Rightarrow$ $x = e^t$ and $dx = e^t\ dt$

$\Rightarrow \quad = \int\left(\log t+\frac{1}{t}\right) e^t\ dt = e^t \log t + C$

$\Rightarrow \quad = e^{\log x} \log\log x + C = x \log\log x + C$

Example : 31

Evaluate : $\int \frac{x^2\,dx}{(x-1)\,(2x+3)}$

Solution

Let I = $\int \frac{x^2 dx}{(x-1)(2x+3)}$

The degree of numerator is not less than the degree of denominator. Hence we divide N by D.

$$\frac{x^2}{(x-1)(2x+3)} = \text{quotient} + \frac{\text{remainder}}{(x-1)(2x+3)} = \frac{1}{2} + \frac{-\frac{1}{2}x+\frac{3}{2}}{(x-1)(2x+3)} = \frac{1}{2} + \frac{1}{2}\;\frac{3-x}{(x-1)(2x+3)}$$

We now split $\frac{3-x}{(x-1)(2x+3)}$ in two partial fractions.

Let f(x) = $\frac{3-x}{(x-1)(2x+3)} = \frac{A}{x-1} + \frac{B}{2x+3}$ where A and B are constants.

Equating the numerators on both sides :

$3 - x = A\,(2x + 3) + B\,(x - 1)$

Now there are two ways to calculate A and B.

1. Comparing the coefficients of like terms

2. Substituting the appropriate values of x.

Method 1 :

Comparing the coefficients of x and x^0, we get :

$-1 = 2A + B$ and $3 = 3A - B$

On solving we have a = 2/5 B = –9/5

Method 2 :

In 3 – x = A(2x + 3) + B (x – 1), pur x = 1, – 3/2

x = 1 ⇒ 3 – 1 = 5A ⇒ A = 2/5

x = –3/2 ⇒ 3 + 3/2 = B (–3/2 – 1) ⇒ B = –9/5

Hence finally we have :

$$f(x) = \frac{\frac{2}{5}}{x-1} + \frac{-\frac{9}{5}}{2x+3}$$

$$\Rightarrow \quad I = \int \left[\frac{1}{2} + \frac{1}{2}f(x)\right] dx$$

$$\Rightarrow \quad I = \frac{x}{2} + \frac{1}{2}\int \frac{\frac{2}{5}}{x-1}\,dx + \frac{1}{2}\int \frac{-\frac{9}{5}}{2x+3}\,dx$$

$$\Rightarrow \quad \frac{x}{2} + \frac{1}{5}\log|x-1| - \frac{9}{20}\log|2x+3| + C$$

Example : 32

Evaluate : $\int \frac{(x-1)\,dx}{(2x+1)(x-2)(x-3)}$

Solution

Let f(x) = $\frac{x-1}{(2x+1)(x-2)(x-3)} = \frac{A}{2x+1} + \frac{B}{x-2} + \frac{C}{x-3}$

$\Rightarrow \quad A = \left.\frac{x-1}{(x-2)(x-3)}\right]_{x=-\frac{1}{2}} = -\frac{6}{35}$

$\Rightarrow \quad B = \left.\frac{x-1}{(2x+1)(x-3)}\right]_{x=2} = -\frac{1}{5}$

$\Rightarrow \quad C = \left.\frac{x-1}{(2x+1)(x-2)}\right]_{x=3} = \frac{2}{7}$

$\Rightarrow \quad \int f(x)\,dx = \frac{-6}{35}\int \frac{dx}{2x+1} - \frac{1}{5}\int \frac{dx}{x-2} + \frac{2}{7}\int \frac{dx}{x-3}$

$= -\frac{3}{35}\log|2x+1| - \frac{1}{5}\log|x-2| + \frac{2}{7}\log|x-3| + C$

Example : 33

Evaluate : $\int \frac{(\cos\theta+1)\sin\theta}{(\cos\theta-1)^2(\cos\theta-3)}$

Solution

Let cos θ = x $\Rightarrow$ – sin θ dθ = dx

$\Rightarrow \quad I = -\int \frac{x+1}{(x-1)^2(x-3)}\,dx$

Let f(x) = $\int \frac{x+1}{(x-1)^2(x-3)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x-3}$

Equating numerator on both sides,

$\Rightarrow \quad x + 1 = A(x-1)(x-3) + B(x-3) + C(x-1)^2$

By taking x = 1, we get B = – 1

By taking x = 3, we get C = 1

Comparing the coefficient of x^2 , we get,

0 = A + C $\Rightarrow$ 0 = A + 1 $\Rightarrow$ A = – 1

$\Rightarrow \quad I = -\int f(x)\,dx = -\left\{\int \frac{-1}{x-1}dx + \int \frac{-1}{(x-1)^2}dx + \int \frac{1}{x-3}dx\right\}$

$\Rightarrow \quad I = \log|x-1| - \frac{1}{x-1} - \log|x-3| + C$

$\Rightarrow \quad I = \log\left|\frac{x-1}{x-3}\right| - \frac{1}{x-1} + C$

$\Rightarrow \quad I = \log\left|\frac{\cos\theta-1}{\cos\theta-3}\right| - \frac{1}{\cos\theta-1} + C$

Example : 34

Evaluate : $\int \frac{dx}{x^3+1}$

Solution

Let $f(x) = \frac{1}{x^3+1} =$

$\Rightarrow$ $f(x) = \frac{1}{(x+1)(x^2-x+1)} = \frac{A}{x+1} + \frac{Bx+C}{x^2-x+1}$

$\Rightarrow$ $1 = a(x^2 - x + 1) + (Bx + C)(x + 1)$

Comparing the coefficients of x^2, x, x^0 :

$0 = A + B,$ $0 = -A + B + C$ $1 = A + C$

$\Rightarrow$ $A = 1/3C = 2/3$ $B = -1/3$

$\Rightarrow$ $f(x) = \frac{\frac{1}{3}}{x+1} + \frac{-\frac{x}{3}+\frac{2}{3}}{x^2-x+1}$

Let $I_1 = \frac{1}{3}\int \frac{dx}{x+1} = \frac{1}{3}\log|x+1| + C_1$

Let $I_2 = \int \frac{-\frac{1}{3}x+\frac{2}{3}}{x^2-x+1}\,dx = \frac{1}{3}\int \frac{2-x}{x^2-x+1}\,dx$

Express the numerator in terms of derivative of denominator.

$\Rightarrow$ $I_2 = -\frac{1}{6}\int \frac{2x-4}{x^2-x+1}\,dx$

$\Rightarrow$ $I_2 = -\frac{1}{6}\int \frac{2x-1}{x^2-x+1}\,dx + \frac{1}{2}\int \frac{dx}{x^2-x+1}$ $\frac{1}{(x+1)(x^2-x+1)}$

$\Rightarrow$ $I_2 = -\frac{1}{6}\log|x^2-x+1| + \frac{1}{2}\int \frac{dx}{\left(x-\frac{1}{2}\right)^2+\frac{3}{4}}$

$\Rightarrow$ $I_2 = -\frac{1}{6}\log|x^2-x+1| + \frac{2}{2\sqrt{3}}\tan^{-1}\left(\frac{x-\frac{1}{2}}{\frac{\sqrt{3}}{2}}\right) + C_2$

$\Rightarrow$ $I_2 = -\frac{1}{6}\log|x^2-x+1| + \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{2x-1}{\sqrt{3}}\right) + C_2$

$\Rightarrow$ $\int \frac{dx}{x^3+1} = \int f(x)\,dx = I_1 + I_2 = \frac{1}{3}\log|x+1| - \frac{1}{6}\log|x^2-x+1| + \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{2x-1}{\sqrt{3}}\right) + C$

$\frac{1}{3}\log\left|\frac{x+1}{\sqrt{x^2-x+1}}\right| + \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{2x-1}{\sqrt{3}}\right) + C$

Example : 35

Evaluate : $\int \frac{x^2 dx}{x^4 - 1}$

Solution

$$\int \frac{x^2 dx}{x^4 - 1} = \int \frac{x^2\ dx}{(x^2 - 1)(x^2 + 1)}$$

$$\frac{x^2}{(x-1)(x+1)(x^2+1)} = \frac{A}{x-1} + \frac{B}{x+1} + \frac{Cx+D}{x^2+1}$$

As the function contains terms of x^2 only, substitute x^2 = t and then make partial fractions

$$\frac{t}{(t-1)(t+1)} = \frac{A}{t-1} + \frac{B}{t+1} \qquad \Rightarrow \qquad t = A\,(t + 1) + B\,(t - 1)$$

Put t = ± 1 to get A = 1/2, B = 1/2

$$\Rightarrow \qquad \frac{t}{(t-1)(t+1)} = \frac{\frac{1}{2}}{t-1} + \frac{\frac{1}{2}}{t+1}$$

Convert t = x^2 again before integrating

$$\Rightarrow \qquad I = \int \frac{x^2 dx}{(x^2-1)(x^2+1)} = \int \frac{1/2}{(x^2-1)}\ dx + \int \frac{1/2}{x^2+1}\ dx$$

$$= \frac{1}{2}\,\frac{1}{2}\ \log\left|\frac{x-1}{x+1}\right| + \frac{1}{2}\ \tan^{-1} x + C$$

Example : 36

Evaluate $\int \frac{dx}{(x+1)\sqrt{x+2}}$

Solution

Let $I = \int \frac{dx}{(x+1)\sqrt{x+2}}$

Substitute : x + 2 = t^2 ⇒ dx = 2t dt

$$\Rightarrow \qquad \int \frac{dx}{x+1\left(\sqrt{x+2}\right)} = \int \frac{2t\ dt}{(t^2-1)\ \sqrt{t^2}} = 2\int \frac{dt}{t^2-1} = \frac{2}{2}\ \log\left|\frac{t-1}{t+1}\right| + C = \log\left|\frac{\sqrt{x+2}-1}{\sqrt{x+2}+1}\right| + C$$

Example : 37

Evaluate : $\int \frac{x}{(x^2-3x+2)\sqrt{x-1}}\ dx$

Solution

Let x – 1 = t^2 ⇒ dx = 2t dt

$$\Rightarrow \qquad I = \int \frac{(t^2+1)}{(t^2+1)^2 - 3(t^2+1) + 2}\ \frac{2t\ dt}{\sqrt{t^2}} = 2\int \frac{(t^2+1)\ dt}{t^4 - t^4}$$

$\Rightarrow \quad I = 2\int \frac{t^2+1}{t^2(t^2-1)}\,dt = \int\left(\frac{2}{t^2-1}-\frac{1}{t^2}\right) dt = 4\int\frac{dt}{t^2-1} - 2\int\frac{dt}{t^2}$

$\Rightarrow \quad I = \frac{4}{2}\log\left|\frac{t-1}{t+1}\right| + \frac{2}{t} + C$

$\Rightarrow \quad I = 2\log\left|\frac{\sqrt{x-1}-1}{\sqrt{x-1}+1}\right| + \frac{2}{\sqrt{x-1}} + C$

Example : 38

Evaluate : $\int\frac{dx}{(x^2+1)\sqrt{x^2+2}}$

Solution

Let $I = \int\frac{dx}{(x^2+1)\sqrt{x^2+2}}$

Substitute : $x = \frac{1}{t} \quad \Rightarrow \quad dx = \frac{1}{t^2}\,dt$

$\Rightarrow \quad I = \int\frac{-\frac{1}{t^2}dt}{\left(\frac{1}{t^2}+1\right)\sqrt{\frac{1}{t^2}+2}} = \int\frac{-t\,dt}{(1+t^2)\sqrt{1+2t^2}}$

Let $\quad 1 + 2t^2 = z^2 \quad \Rightarrow \quad 4t\,dt = 2z\,dz$

$\Rightarrow \quad I = \frac{-1}{2}\int\frac{z\,dz}{\left(1+\frac{z^2-1}{2}\right)\sqrt{z^2}} = \int\frac{dz}{z^2+1} = -\tan^{-1} z + C$

$\Rightarrow \quad I = -\tan^{-1}\sqrt{1+2t^2} + C = -\tan^{-1}\sqrt{1+\frac{2}{x^2}} + C$

Example : 39

Evaluate : $\int\frac{dx}{(x+2)\sqrt{x^2+6x+7}}$

Solution

Let $I = \int\frac{dx}{(x+2)\sqrt{x^2+6x+7}}$

Substitute : $x + 2 = \frac{1}{t} \quad \Rightarrow \quad dx = -\frac{1}{t^2}$

$\Rightarrow \quad x^2 + 6x + 7 = \left(\frac{1}{t}-2\right)^2 + 6\left(\frac{1}{t}-2\right) + 7 = \frac{1+2t-t^2}{t^2}$

$\Rightarrow \quad I = \int \frac{dt}{\sqrt{2-(t-1)^2}} = -\sin^{-1}\left(\frac{t-1}{\sqrt{2}}\right) + C$

$\Rightarrow \quad I = \sin^{-1}\left[\frac{x+1}{(x+2)\sqrt{2}}\right] + C$

Example : 40

Evaluate : $\int \frac{dx}{\sqrt[3]{x+1}+\sqrt{x+1}}$

Solution

Let $\quad I = \int \frac{dx}{\sqrt[3]{x+1}+\sqrt{x+1}} \quad \Rightarrow \quad I = \int \frac{dx}{(x+1)^{1/3}+(x+1)^{1/2}}$

The least common multiple of 2 and 3 is 6

So substitute $x + 1 = t^6 \quad \Rightarrow \quad dx = 6t^5\, dt$

$\Rightarrow \quad I = \int \frac{6t^5 dt}{t^2+t^3} = 6 \int \frac{t^3 dt}{1+t}$

$\Rightarrow \quad I = 6 \int \left(t^2 - t + 1 - \frac{1}{1+t}\right) dt$

$\Rightarrow \quad I = 6\left(\frac{t^3}{3} - \frac{t^2}{2} + t - \log(t+1)\right) + C$

On substituting $t = (1 + x)^{1/6}$, we get

$I = 6\left(\frac{(1+x)^{1/2}}{3} - \frac{(1+x)^{1/3}}{2} + (1+x)^{1/6} - \log\left((x+1)^{1/6}+1\right)\right) + C$

Example : 41

Evaluate : $\int x^{13/2}(1+x^{5/2})^{1/2}\, dx$

Solution

Let $\quad I = \int x^{13/2}(1+x^{5/2})^{1/2}\, dx$

Comparing with integral of type 5.6, we can see that p = 1/2 which is not an integer.

So this integral does not belong to type 5.6 (i).

Check the sign of (m + 1)/n

$\frac{m+1}{n} = \frac{\frac{13}{2}+1}{\frac{5}{2}} = \frac{15}{5} = 3 \quad \Rightarrow \quad$ (m + 1)/n is an integer. So this integral belongs to type 5.6 (ii)

To solve this integral, substitute $1 + x^{5/2} = t^2$

$\Rightarrow \quad 5/2\, x^{3/2}\, dx = 2t\, dt$

$\Rightarrow \quad I = \frac{2}{5} \int (t^2-1)^2 (t^2)^{1/2}\, 2t\, dt$

$\Rightarrow \quad I = \frac{4}{5} \int t^2 (t^2-1)^2\, dt$

$\Rightarrow \quad I = \frac{4}{5}\int t^6 + t^2 - 2t^4\, dt$

$\Rightarrow \quad I = \frac{4}{5}\left(\frac{t^7}{7} + \frac{t^3}{3} - 2\frac{t^5}{5}\right) + C$

On substituting $t = (1 + x^{5/2})^{1/2}$, we get

$$I = \frac{4}{5}\left(\frac{(1+x^{5/2})^{7/2}}{7} + \frac{(1+x^{5/2})^{3/2}}{3} - \frac{2(1+x^{5/2})^{5/2}}{5}\right) + C$$

Example : 42

Evaluate : **(1)** $\int \frac{dx}{(2ax + x^2)^{3/2}}$ **(2)** $\int \frac{\sqrt{x^2+1}}{x^4}\, dx$

Solution

(1) Let $I = \int \frac{dx}{(2ax + x^2)^{3/2}} \Rightarrow I = \int \frac{dx}{[(x+a)^2 - a^2]^{3/2}}$

Put $x + a = a \sec \theta \Rightarrow dx = a \sin \theta \tan \theta\, d\theta$

On substituting in I, we get

$I = \int \frac{a \sec\theta \tan\theta d\theta}{(a^2 \sec^2\theta - a^2)^{3/2}} = \int \frac{2\sec\theta \tan\theta d\theta}{a^3 \tan^3\theta}$

$\Rightarrow \quad I = \frac{1}{a^2}\int \sec\theta \cot^2\theta d\theta = \frac{1}{a^2}\int \frac{\cos\theta}{\sin^2\theta} d\theta$

$\Rightarrow \quad I = \frac{1}{a^2}\int \frac{d\,(\sin\theta)}{\sin^2\theta} d\theta = -\frac{1}{a^2 \sin\theta} + C$

$\Rightarrow \quad I = -\frac{1}{a^2}\ \frac{x+a}{\sqrt{x^2+2ax}} + C$

(2) Let $I = \int \frac{\sqrt{x^2+1}}{x^4}\, dx$

Put x tan θ $\Rightarrow$ $dx = \sec^2\theta\, d\theta$

On substituting x and dx in I, we get

$I = \int \frac{\sqrt{\tan^2\theta + 1}}{\tan^4\theta} \sec^2\theta\ d\theta = \int \frac{\sec^3\theta d\theta}{\tan^4\theta}$

$\Rightarrow \quad I = \int \frac{\cos\theta}{\sin^4\theta} d\theta = \int \frac{d(\sin\theta)d\theta}{\sin^4\theta} \Rightarrow I = -\frac{1}{3\sin^3\theta} + C$

On substituting value of sin θ in terms of x, we get $I = -\frac{1}{3}\ \frac{(1+x^2)^{3/2}}{x^3} + C$

Example : 43

Find the reduction formula for $\int \sin^n x\, dx$

Solution

Let $I_0 = \int \sin^n x\, dx = \int \sin^{n-1} x \,.\, \sin x\, dx$

Apply by parts taking $\sin^{n-1}$ x as first part and sin x as second part.

$\Rightarrow \quad I_n = \sin^{n-1} x \,.\, (-\cos x) + \int (n-1)\sin^{n-2} x \cos^2 x\, dx$

$= -\cos x \sin^{n-1} x + (n - 1) \int \sin^{n-2} x(1-\sin^2 x)\, dx$

$= -\cos x \sin^{n-1} x + (n-1) \int \sin^{n-2} x\, dx - (n-1)\int \sin^n x\, dx$

$\Rightarrow \quad I_n = -\cos x \sin^{n-1}x + (n-1)\, I_{n-2} - (n-1)\, I_n$

$\Rightarrow \quad nI_n = -\cos x \,.\, \sin^{n-1} x + (n-1)\, I_{n-2}$;

$$I_n = -\frac{\cos x \sin^{n-1} x}{n} + \frac{n-1}{n} I_{n-2}$$

Example : 44

Find the reduction formula for $\int \tan^n x\, dx$.

If $I_n = \int \tan^n x\, dx$, to prove that $(n-1)(I_n + I_{n-2}) = \tan^{n-1} x$.

Solution

Here $I_n = \int \tan^n x\, dx \quad \int \tan^{n-2} x \tan^2 x\, dx$

$= \int \tan^{n-2} x(\sec^2 x - 1)\, dx$

$= \int \tan^{n-2} x \sec^2 x\, dx - \int \tan^{n-2} x\, dx$

$= \int \tan^{n-2} x \sec^2 x - I_{n-2}$

$$\Rightarrow \quad I_n + I_{n-2} = \frac{\tan^{n-1} x}{n-1}$$

Hence $(n-1)(I_n + I_{n-2}) = \tan^{n-1} x$.

Example : 45

Find reduction formula for $\int \sec^n x\, dx$

Solution

Let $I_n = \int \sec^n x$

$\Rightarrow \quad I_n = \int \sec^{n-2} x \sec^2 x\, dx$

Apply by parts taking $\sec^{n-2}$ x as the first part and $\sec^2$x as the second part

$$\Rightarrow \quad I_n = \sec^{n-2} x \int \sec^3 x\, dx - \int \left[\frac{d}{dx}(\sec^{n-2} x) \int \sec^2 x\, dx\right] dx$$

$\Rightarrow \quad I_n = \sec^{n-2} x \tan x - \int (n-2) \sec^{n-3} x \sec x \tan x \tan x\, dx$

$\Rightarrow \quad I_n = \sec^{n-2} x \tan x - (n-2) \int \sec^{n-2} x (\sec^2 x - 1)\, dx$

$\Rightarrow \quad I_n + (n-2)\, I_n = \sec^{n-2} x \tan x + (n-2) \int \sec^{n-2} x\, dx$

$\Rightarrow \quad (n-1)\, I_n = \sec^{n-2} x \tan x + (n-2)\, I_{n-2}$

Hence $\int \sec^n x \, dx = \dfrac{\sec^{n-2} x \tan x}{n-1} + \dfrac{n-2}{n-1} \int \sec^{n-2} x \, dx$

This is the required reduction formula for $\int \sec^n x \, dx$

Example : 46

Find the reduction formula for $\int e^{ax} \cos^n x \, dx$

Solution

Let $\quad I_n = \int e^{ax} \cos^n x \, dx$

Apply by parts taking $\cos^n x$ as the first part and e^{ax} the second part

$$\Rightarrow \quad I_n = \cos^n x \int e^{ax} \, dx - \int \left[\frac{d}{dx}(\cos^n x) \int e^{ax} dx \right] dx$$

$$\Rightarrow \quad I_n = \frac{e^{ax}}{a} \cos^n x - \int n \cos^{n-1} x \ x \ (-\sin x) \ \frac{e^{ax}}{a} \ dx$$

$$\Rightarrow \quad I_n = \frac{1}{a} e^{ax} \cos^n x + \frac{n}{a} \int (\cos^{n-1} x \sin x) \ e^{ax} \, dx$$

Apply by parts again taking $\cos^{n-1} x \sin x$ as first part and e^{ax} as second part

$$\Rightarrow \quad I_n = \frac{1}{a} e^{ax} \cos^n x + \frac{n}{a} (\cos^{n-1} x \sin x) \int e^{ax} - \frac{n}{a} \int \left[\frac{d}{dx}(\cos^{n-1} x \sin x) \int e^{ax} \, dx \right] dx$$

$$\Rightarrow \quad I_n = \frac{1}{a} e^{ax} \cos^n x + \frac{n}{a} \cos^{n-1} x \sin x \ \frac{e^{ax}}{a} - \frac{n}{a} \int [-(n-1) \cos^{n-2} x \sin^2 x + \cos^{n-1} x \, . \cos x] \ \frac{e^{ax}}{a} \ dx$$

$$\Rightarrow \quad I_n = \frac{1}{a} e^{ax} \cos^n x + \frac{n}{a^2} e^{ax} \cos^{n-1} x \sin x + \frac{n(n-1)}{a^2} \int e^{ax} \cos^{n-2} x \, (1 - \cos^2 x) \, dx - \frac{n}{a^2} \int e^{ax} \cos^n x \, dx$$

$$\Rightarrow \quad I_n = \frac{1}{a} e^{ax} \cos^n x + \frac{n}{a^2} e^{ax} \cos^{n-1} x \sin x + \frac{n(n-1)}{a^2} I_{n-2} - \frac{n^2}{a^2} I_n$$

$$\Rightarrow \quad I_n = \left(1 + \frac{n^2}{a^2}\right) I_n = \frac{1}{a^2} e^{ax} (a \cos x + n \sin x) \cos^{n-1} x + \frac{n(n-1)}{a^2} I_{n-2}$$

Hence $\int e^{ax} \cos^n x \, dx = e^{ax} \left(\dfrac{a \cos x + n \sin x}{a^2 + n^2} \right) \cos^{n-1} x + \dfrac{n(n-1)}{a^2 + n^2} \int e^{ax} \cos^{n-2} x \, dx$

This is the required reduction formula.

Example : 47

Find the reduction formula for $\int \cos^m x \ \sin nx \, dx$

Solution

Let $I_{m,n} = \int \cos^m x \ \sin nx \, dx$

Apply by parts taking $\cos^m x$ as the first part and sin nx as the second part.

$$\Rightarrow \qquad I_{m,n} = \cos^m x \left(-\frac{\cos nx}{n}\right) - \int m \cos^{m-1} (-\sin x) \left(-\frac{\cos nx}{n}\right) dx$$

$$\Rightarrow \qquad I_{m,n} = -\frac{\cos^m x \cos nx}{n} - \frac{m}{n} \int \cos^{m-1} x \ (\sin x \cos nx) \, dx$$

Now sin (n – 1) x = sin nx cos x – cos nx sin x or cos nx sin x = sin nx cos x – sin (n – 1) x

$$\Rightarrow \qquad I_{m,n} = -\frac{\cos^m x \cos nx}{n} - \frac{m}{n} \int \cos^{m-1} x \ [\sin nx \cos x - \sin (n-1) x] \, dx$$

$$\Rightarrow \qquad I_{m,n} = -\frac{\cos^m x \cos nx}{n} - \frac{m}{n} \int \cos^m x \sin nx \, dx + \frac{m}{n} \int \cos^{m-1} x \sin (n-1) x \, dx$$

$$\Rightarrow \qquad \left[1 + \frac{m}{n}\right] I_{m,n} = -\frac{\cos^m x \cos nx}{n} + \frac{m}{n} I_{m-1,n-1}$$

$$\Rightarrow \qquad I_{m,n} = \frac{m}{m+n} I_{m-1,n-1} - \frac{\cos^m x \cos nx}{m+n}$$