

Example : 1

Find the domain and the range of the following functions

(a) $y = \sqrt{1-x^2}$

(b) $y = 2 \sin x$

(c) $y = \frac{1}{x-2}$

Solution

(a) For domain : $1 - x^2 \geq 0$
 $\Rightarrow x^2 \leq 1$
 $\Rightarrow -1 \leq x \leq 1$

Hence the domain is x set $[-1, 1]$.

For range : As $-1 \leq x \leq 1$
 $\Rightarrow 0 \leq x^2 \leq 1$
 $\Rightarrow 0 \leq 1 - x^2 \leq 1$
 $\Rightarrow 0 \leq \sqrt{1-x^2} \leq 1$
 $\Rightarrow 0 \leq y \leq 1$

Hence the range is set $[0, 1]$

(b) $y = 2 \sin x$

For domain : $x \in \mathbb{R}$ i.e. $x \in (-\infty, \infty)$

For range : $-1 \leq \sin x \leq 1$
 $-2 \leq 2 \sin x \leq 2$
 $-2 \leq y \leq 2$

Hence the range is $y \in [-2, 2]$

(c) As denominator cannot be zero, x can not be equal to 2
 domain is $x \in \mathbb{R} - \{2\}$

i.e. $x \in (-\infty, 2) \cup (2, \infty)$

Range : As y can never become zero, the range is $y \in \mathbb{R} - \{0\}$

i.e. $y \in (-\infty, 0) \cup (0, \infty)$

Example : 2

Find the domain of the following functions :

(a) $\sqrt{3-x} + \frac{1}{\log_{10} x}$

(b) $\frac{1}{x+|x|}$

(c) $\sqrt{1-\log_{10} x}$

Solution

(a) $\sqrt{3-x}$ is defined if $3-x \geq 0$
 $\Rightarrow x \leq 3$ (i)

$\frac{1}{\log_{10} x}$ is defined if $x > 0$ and $x \neq 1$

$\Rightarrow x > 0 - \{1\}$ (ii)

Combining (i) and (ii), set of domain is :

$x \in (0, 1) \cup (1, 3]$

(b) $f(x)$ is defined if : $x + |x| \neq 0$

$\Rightarrow |x| \neq -x \Rightarrow x > 0$

Hence domain is $x \in (0, \infty)$

(c) $f(x)$ is defined if

$1 - \log_{10} x \geq 0$ and $x > 0$

$\Rightarrow \log_{10} x \leq 1$ and $x > 0$

$\Rightarrow x \leq 10$ and $x > 0$

\Rightarrow domain is $x \in (0, 10]$

Example : 3

(Using factorisation) Evaluate the following limits :

(a) $\lim_{x \rightarrow 2} \frac{x^3 - 2x - 4}{x^2 - 3x + 2}$

(b) $\lim_{x \rightarrow a} \frac{x^3 - a^3}{x^2 - ax}$

(c) $\lim_{x \rightarrow 5} \frac{x^4 - 625}{x^3 - 125}$

Solution

$$(a) \quad \lim_{x \rightarrow 2} \frac{x^3 - 2x - 4}{x^2 - 3x + 2} = \lim_{x \rightarrow 2} \frac{(x-2)(x^2 + 2x + 2)}{(x-2)(x-1)} = \lim_{x \rightarrow 2} \frac{x^2 + 2x + 2}{x-1} = 10$$

$$(b) \quad \lim_{x \rightarrow a} \frac{x^3 - a^3}{x^2 - ax} = \lim_{x \rightarrow a} \frac{(x-a)(x^2 + ax + a^2)}{x(x-a)} = \lim_{x \rightarrow a} \frac{x^2 + ax + a^2}{x} = 3a$$

$$(c) \quad \lim_{x \rightarrow 5} \frac{x^4 - 625}{x^3 - 125} = \lim_{x \rightarrow 5} \frac{x^4 - 5^4}{x^3 - 5^3} = \frac{4.5^3}{3.5^2} = \frac{20}{3} \quad [\text{using section 2.2 (ix)}]$$

Example : 4

(Using rationalisation) Evaluate the following limits :

$$(a) \quad \lim_{x \rightarrow 3} \frac{\sqrt{3x+7} - 4}{\sqrt{x+1} - 2} \quad (b) \quad \lim_{x \rightarrow a} \frac{\sqrt{a+2x} - \sqrt{3x}}{\sqrt{3a+x} - 2\sqrt{x}} \quad (c) \quad \lim_{x \rightarrow 1} \frac{\sqrt[3]{x} - 1}{x^2 - 1}$$

Solution

$$(a) \quad \lim_{x \rightarrow 3} \frac{\sqrt{3x+7} - 4}{\sqrt{x+1} - 2}$$

Rationalising the numerator and denominator,

$$= \lim_{x \rightarrow 3} \frac{3x+7-16}{x+1-4} \left(\frac{\sqrt{x+1}+2}{\sqrt{3x+7}+4} \right)$$

$$= \lim_{x \rightarrow 3} \frac{3(x-3)}{x-3} \left(\frac{\sqrt{x+1}+2}{\sqrt{3x+7}+4} \right)$$

$$= 3 \lim_{x \rightarrow 3} \frac{\sqrt{x+1}+2}{\sqrt{3x+7}+4} = 3 \left(\frac{2+2}{4+4} \right) = \frac{3}{2}$$

(b) Rationalising numerator and denominator we get,

$$= \lim_{x \rightarrow a} \frac{a+2x-3x}{3a+x-4x} \left(\frac{\sqrt{3a+x}+2\sqrt{x}}{\sqrt{a+2x}+\sqrt{3x}} \right)$$

$$= \lim_{x \rightarrow a} \frac{a-x}{3(a-x)} \left(\frac{\sqrt{3a+x}+2\sqrt{x}}{\sqrt{a+2x}+\sqrt{3x}} \right) = \frac{1}{3} \left(\frac{2\sqrt{a}+2\sqrt{a}}{\sqrt{3a}+\sqrt{3a}} \right) = \frac{2}{3\sqrt{3}}$$

$$(c) \quad \lim_{x \rightarrow 1} \frac{x^{\frac{1}{3}} - 1}{x^2 - 1} = \lim_{x \rightarrow 1} \frac{\left(\frac{1}{x^3} - 1 \right) \left(\frac{2}{x^3} + \frac{1}{x^3} + 1 \right)}{(x^2 - 1) \left(\frac{2}{x^3} + \frac{1}{x^3} + 1 \right)}$$

$$\lim_{x \rightarrow 1} \frac{(x-1)}{(x-1)(x+1) \left(\frac{2}{x^3} + \frac{1}{x^3} + 1 \right)} = \frac{1}{6}$$

Example : 5

($x \rightarrow \infty$ type problems) Evaluate the following limits :

$$(a) \lim_{x \rightarrow \infty} \frac{x^3 - 2x^2 + 3x + 1}{5x^3 + 7x + 2} \quad (b) \lim_{n \rightarrow \infty} \frac{1^2 + 2^2 + 3^2 + \dots + n^2}{n^3} \quad (c) \lim_{x \rightarrow \infty} \frac{\sqrt{x^2 + 3x + 1} + 5x}{1 + 4x}$$

Solution

In these type of problems, divide numerator and denominator by highest power of x .

(a) Dividing numerator and denominator by x^3

$$= \lim_{x \rightarrow \infty} \frac{1 - \frac{2}{x} + \frac{3}{x^2} + \frac{1}{x^3}}{5 + \frac{7}{x^2} + \frac{2}{x^3}} = \frac{1}{5}$$

[because as $x \rightarrow \infty$, $\frac{1}{x}$, $\frac{1}{x^2}$, $\frac{1}{x^3}$ $\rightarrow 0$]

$$(b) \lim_{n \rightarrow \infty} \frac{1^2 + 2^2 + 3^2 + \dots + n^2}{n^3}$$

$$= \lim_{n \rightarrow \infty} \frac{n(n+1)(2n+1)}{6n^3}$$

$$= \frac{1}{6} \lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right) \left(2 + \frac{1}{n}\right)$$

$$= \frac{1}{6} (1 + 0) (2 + 0) = \frac{1}{3}$$

(c) The highest power of x is 1. Hence divide the numerator and denominator by x .

$$\lim_{x \rightarrow \infty} \frac{\sqrt{x^2 + 3x + 1} + 5x}{1 + 4x}$$

$$= \lim_{x \rightarrow \infty} \frac{\sqrt{1 + \frac{3}{x} + \frac{1}{x^2}} + 5}{\frac{1}{x} + 4} = \frac{\sqrt{1+5}}{0+4} = \frac{3}{2}$$

Example : 6

($\infty - \infty$ form) Evaluate the following limits :

$$(a) \lim_{x \rightarrow \frac{\pi}{2}} (\sec x - \tan x) \quad (b) \lim_{x \rightarrow \infty} \left(\sqrt{(x+2a)(2x+a)} - x\sqrt{2} \right) \quad (c) \lim_{x \rightarrow \infty} \left(x - \sqrt{x^2 + x} \right)$$

Solution

$$(a) \lim_{x \rightarrow \frac{\pi}{2}} (\sec x - \tan x) = \lim_{x \rightarrow \frac{\pi}{2}} \frac{\sec^2 x - \tan^2 x}{\sec x + \tan x}$$

$$\lim_{x \rightarrow \frac{\pi}{2}} \frac{1}{\sec x + \tan x} = 0$$

$$(b) \lim_{x \rightarrow \infty} \left(\sqrt{(x+2a)(2x+a)} - x\sqrt{2} \right)$$

Rationalising the expression, we get

$$\lim_{x \rightarrow \infty} \frac{(x+2a)(2x+a) - 2x^2}{\sqrt{(x+2a)(2x+a)} + x\sqrt{2}}$$

$$= \lim_{x \rightarrow \infty} \frac{5ax + 2a^2}{\sqrt{2x^2 + 5ax + 2a^2} + x\sqrt{2}}$$

Dividing numerator and denominator by x, we get

$$= \lim_{x \rightarrow \infty} \frac{5a + \frac{2a^2}{x}}{\sqrt{2 + \frac{5a}{x} + \frac{2a^2}{x^2}} + \sqrt{2}} = \frac{5a}{2\sqrt{2}}$$

(c) $\lim_{x \rightarrow \infty} (x - \sqrt{x^2 + x})$

On rationalising the expression, we get

$$= \lim_{x \rightarrow \infty} \frac{x^2 - (x^2 + x)}{x + \sqrt{x^2 + x}} = \lim_{x \rightarrow \infty} \frac{-x}{x + \sqrt{x^2 + x}}$$

Divide by the highest power of x i.e. x^1

$$= \lim_{x \rightarrow \infty} \frac{-1}{1 + \sqrt{1 + \frac{1}{x}}} = \frac{-1}{1+1} = -\frac{1}{2}$$

Example : 7

(using $\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1$) Evaluate the following limits :

(a) $\lim_{x \rightarrow \frac{\pi}{3}} \frac{\tan x - \sqrt{3}}{9x^2 - \pi^2}$

(b) $\lim_{x \rightarrow \frac{\pi}{2}} \frac{\sin(\cos x) \cos x}{\sin x - \operatorname{cosec} x}$

(c) $\lim_{x \rightarrow a} \frac{\cos x - \cos a}{x - a}$

(d) $\lim_{x \rightarrow a} \frac{a \sin x - x \sin a}{ax^2 - a^2x}$

Solution

(a) $\lim_{x \rightarrow \frac{\pi}{3}} \frac{\tan x - \sqrt{3}}{9x^2 - \pi^2} = \lim_{x \rightarrow \frac{\pi}{3}} \frac{\tan x - \frac{\pi}{3}}{9x^2 - \pi^2}$

Using $\tan A - \tan B = \frac{\sin(A - B)}{\cos A \cos B}$ we get,

$$\lim_{x \rightarrow \frac{\pi}{3}} \frac{\sin\left(x - \frac{\pi}{3}\right)}{\cos x \cos \frac{\pi}{3} (3x - \pi)(3x + \pi)} = \frac{1}{3} \frac{1}{\cos \frac{\pi}{3} \cos \frac{\pi}{3} (\pi + \pi)} \left(\text{using } \lim_{\theta \rightarrow 0} \frac{\sin \theta}{\theta} = 1 \right) = \frac{2}{3\pi}$$

(b) $\lim_{x \rightarrow \frac{\pi}{2}} \frac{\sin(\cos x) \cos x}{\sin x - \operatorname{cosec} x} = \lim_{x \rightarrow \frac{\pi}{2}} \frac{\sin(\cos x)}{\cos x}$

$$\lim_{x \rightarrow \frac{\pi}{2}} \frac{\cos^2 x}{\sin x - \operatorname{cosec} x} = 1 \times \lim_{x \rightarrow \frac{\pi}{2}} \frac{\cos^2 x \sin x}{\sin^2 x - 1} \left(\text{using } \lim_{\theta \rightarrow 0} \frac{\sin \theta}{\theta} = 1 \right)$$

$$= - \lim_{x \rightarrow \frac{\pi}{2}} (\sin x) = -1$$

$$(c) \quad \lim_{x \rightarrow a} \frac{\cos x - \cos a}{x - a} \lim_{x \rightarrow a} \frac{-2 \sin\left(\frac{x+a}{2}\right) \sin\left(\frac{x-a}{2}\right)}{2 \times \left(\frac{x-a}{2}\right)}$$

$$= - \lim_{x \rightarrow a} \sin\left(\frac{x+a}{2}\right) \lim_{x \rightarrow a} \frac{\sin\left(\frac{x-a}{2}\right)}{\left(\frac{x-a}{2}\right)} = - \sin a$$

$$(d) \quad \lim_{x \rightarrow a} \frac{a \sin x - x \sin a}{ax^2 - a^2x}$$

$$= \lim_{x \rightarrow a} \frac{a \sin x - x \sin x + x \sin x - x \sin a}{ax(x-a)} = \lim_{x \rightarrow a} \frac{(a-x) \sin x + x(\sin x - \sin a)}{ax(x-a)}$$

$$= \lim_{x \rightarrow a} \frac{(a-x) \sin x}{ax(x-a)} + \lim_{x \rightarrow a} \frac{\sin x - \sin a}{a(x-a)} = -\frac{\sin a}{a^2} + \lim_{x \rightarrow a} \frac{2 \cos\left(\frac{x+a}{2}\right) \left[\frac{\sin\left(\frac{x-a}{2}\right)}{\frac{(x-a)}{2}}\right]}{2a}$$

$$= \frac{\sin a}{a^2} + \frac{\cos a}{a}$$

Example : 8

$\left(\text{using } \lim_{x \rightarrow 0} \frac{a^x - 1}{x} = \log a\right)$ Evaluate the following limits :

$$(a) \quad \lim_{x \rightarrow 1} \frac{2^x - 2}{x - 1} \qquad (b) \quad \lim_{x \rightarrow a} \frac{e^{\sqrt{x}} - e^{\sqrt{a}}}{x - a}$$

$$(c) \quad \lim_{x \rightarrow 0} \frac{6^x - 2^x - 3^x + 1}{\sin^2 x} \qquad (d) \quad \lim_{x \rightarrow 0} \frac{3^x - 5^x}{x}$$

Solution

$$(a) \quad \lim_{x \rightarrow 1} \frac{2^x - 2}{x - 1} = 2 \lim_{x \rightarrow 1} \frac{2^{x-1} - 1}{x - 1} = 2 \log 2$$

$$(b) \quad \lim_{x \rightarrow a} \frac{e^{\sqrt{x}} - e^{\sqrt{a}}}{x - a} = \lim_{x \rightarrow a} \frac{e^{\sqrt{a}} (e^{\sqrt{x} - \sqrt{a}} - 1)}{x - a}$$

$$= e^{\sqrt{a}} \lim_{x \rightarrow a} \frac{e^{\sqrt{x} - \sqrt{a}} - 1}{\sqrt{x} - \sqrt{a}} \lim_{x \rightarrow a} \frac{\sqrt{x} - \sqrt{a}}{x - a}$$

$$= e^{\sqrt{a}} (1) \lim_{x \rightarrow a} \frac{(x-a)}{(x-a)(\sqrt{x} + \sqrt{a})} = \frac{e^{\sqrt{a}}}{2\sqrt{a}}$$

$$\begin{aligned}
 \text{(c)} \quad & \lim_{x \rightarrow 0} \frac{6^x - 2^x - 3^x + 1}{\sin^2 x} \\
 &= \lim_{x \rightarrow 0} \frac{(2^x - 1)3^x - 1}{x^2} \frac{x^2}{\sin^2 x} \\
 &= \lim_{x \rightarrow 0} \frac{2^x - 1}{x} \lim_{x \rightarrow 0} \frac{3^x - 1}{x} \lim_{x \rightarrow 0} \left(\frac{x}{\sin x} \right)^2 = \log_e^2 \log_e^2 \\
 \text{(d)} \quad & \lim_{x \rightarrow 0} \frac{3^x - 5^x}{x} = \lim_{x \rightarrow 0} \left[\frac{3^x - 1}{x} - \frac{5^x - 1}{x} \right] = \log 3 - \log 5 = \log \frac{3}{5}
 \end{aligned}$$

Example : 9

$\left[\text{using } \lim_{x \rightarrow 0} (1+x)^{\frac{1}{x}} = e \right]$ Evaluate the following limits :

$$\text{(a)} \quad \lim_{x \rightarrow 0} (1-2x)^{\frac{1}{x}} \qquad \text{(b)} \quad \lim_{x \rightarrow 1} x^{\cot \pi x}$$

Solution

$$\text{(a)} \quad \lim_{x \rightarrow 0} (1-2x)^{\frac{1}{x}} = \lim_{x \rightarrow 0} \left((1-2x)^{\frac{-1}{2x}} \right)^{-2}$$

$$\begin{aligned}
 \text{(b)} \quad \lim_{x \rightarrow 1} x^{\cot \pi x} &= \lim_{x \rightarrow 1} \left[1 + x - 1 \right]^{\frac{1}{x-1} (x-1) \cot \pi x} \\
 &= e^{\lim_{x \rightarrow 1} (x-1) \cot \pi x} = e^{\lim_{x \rightarrow 1} \frac{1-x}{\tan(\pi - \pi x)}} = e^{\lim_{x \rightarrow 1} \frac{1}{\pi} \frac{\pi - \pi x}{\tan(\pi - \pi x)}} = e^{\frac{1}{\pi}} \quad \left(\text{using } \lim_{\theta \rightarrow 0} \frac{\tan \theta}{\theta} = 1 \right)
 \end{aligned}$$

Example : 10

Show that the limit of :

$$f(x) = \begin{cases} 2x - 1 & ; x \leq 1 \\ x & ; x > 1 \end{cases} \text{ at } x = 1 \text{ exists}$$

Solution

$$\text{Left hand limit} = \lim_{x \rightarrow 1^-} f(x) = \lim_{x \rightarrow 1^-} (2x - 1) = 2 - 1 = 1$$

(we use $f(x) = 2x - 1$ \because while calculating limit at $x = 1$, we approach $x = 1$ from LHS i.e. $x < 1$)

$$\text{Right hand limit} = \lim_{x \rightarrow 1^+} f(x) = \lim_{x \rightarrow 1^+} (x) = 1$$

\Rightarrow L.H.L. = R.H.L. = 1. Hence limit exists

Example : 11

Find whether the following limits exist or not :

(a) $\lim_{x \rightarrow 0} \sin \frac{1}{x}$

(b) $\lim_{x \rightarrow 0} x \sin \frac{1}{x}$

Solution

(a) As $x \rightarrow 0$, $\frac{1}{x} \rightarrow \infty$.

As the angle θ approaches ∞ , $\sin \theta$ oscillates by taking values between -1 and $+1$.

Hence $\lim_{x \rightarrow 0} \sin \frac{1}{x}$ is not a well defined finite number.

\Rightarrow limit does not exist

(b) $\lim_{x \rightarrow 0} x \sin \frac{1}{x} = \lim_{x \rightarrow 0} x \lim_{x \rightarrow 0} \sin \frac{1}{x}$

$$= 0 \times (\text{some quantity between } -1 \text{ and } +1) = 0$$

It can be easily seen that $\lim_{x \rightarrow 0^+} x \sin \frac{1}{x} = \lim_{x \rightarrow 0^-} x \sin \frac{1}{x} = 0$

Hence the limit exists and is equal to zero (0)

Example : 12

Comment on the following limits :

(a) $\lim_{x \rightarrow 1} [x - 3]$

(b) $\lim_{x \rightarrow 0} \frac{|x|}{x}$

Solution

(a) Right Hand limit = $\lim_{x \rightarrow 1^+} [x - 3]$

$$= \lim_{h \rightarrow 0} [1 + h - 3] = \lim_{h \rightarrow 0} [h - 2]$$

$$= -2 \text{ (because } h - 2 \text{ is between } -1 \text{ and } -2)$$

Left hand limit = $\lim_{x \rightarrow 1^-} [x - 3]$

$$= \lim_{h \rightarrow 0} [1 - h - 3] = \lim_{h \rightarrow 0} [-2 - h]$$

$$= -3 \text{ (because } -h - 2 \text{ is between } -2 \text{ and } -3)$$

Hence R.H.L. \neq L.H.L.

\Rightarrow limit does not exist.

(b) Left hand limit = $\lim_{x \rightarrow 0^-} \frac{|x|}{x} = \lim_{x \rightarrow 0^-} \frac{-x}{x} = -1$

Right hand limit = $\lim_{x \rightarrow 0^+} \frac{|x|}{x} = \lim_{x \rightarrow 0^+} \frac{x}{x} = +1$

Hence R.H.L. \neq L.H.L.

\Rightarrow limit does not exist

Example : 13

Find a and b so that the function :

$$f(x) = \begin{cases} x + a\sqrt{2} \sin x & ; 0 \leq x < \frac{\pi}{4} \\ 2x \cot x + b & ; \frac{\pi}{4} \leq x \leq \frac{\pi}{2} \\ a \cos 2x - b \sin x & ; \frac{\pi}{2} < x \leq \pi \end{cases}$$

is continuous for $x \in [0, \pi]$

Solution

At $x = \pi/4$

$$\text{Left hand limit} = \lim_{x \rightarrow \frac{\pi}{4}^-} f(x) = \lim_{x \rightarrow \frac{\pi}{4}^-} (x + a\sqrt{2} \sin x) = \frac{\pi}{4} + a$$

$$\text{Right hand limit} = \lim_{x \rightarrow \frac{\pi}{4}^+} f(x) = \lim_{x \rightarrow \frac{\pi}{4}^+} (2x \cot x + b) = \frac{\pi}{2} + b$$

$$f\left(\frac{\pi}{4}\right) = 2\left(\frac{\pi}{4}\right) \cot \frac{\pi}{4} + b = \frac{\pi}{2} + b$$

for continuity, these three must be equal

$$\Rightarrow \frac{\pi}{4} + a = \frac{\pi}{2} + b \Rightarrow a - b = \frac{\pi}{4} \quad \dots\dots\dots(i)$$

At $x = \pi/2$

$$\text{Left hand limit} = \lim_{x \rightarrow \frac{\pi}{2}^-} (2x \cot x + b) = 0 + b = b$$

$$\text{Right hand limit} = \lim_{x \rightarrow \frac{\pi}{2}^+} (a \cos 2x - b \sin x) = -a - b$$

$$f\left(\frac{\pi}{2}\right) = 0 + b$$

for continuity, $b = -a - b$

$$\Rightarrow a + 2b = 0 \quad \dots\dots\dots(ii)$$

Solving (i) and (ii) for a and b, we get : $b = -\frac{\pi}{12}$, $a = \frac{\pi}{6}$

Example : 14

A function $f(x)$ satisfies the following property $f(x + y) = f(x) f(y)$. Show that the function is continuous for all values of x if it is continuous at $x = 1$

Solution

As the function is continuous at $x = 1$, we have

$$\lim_{x \rightarrow 1^-} f(x) = \lim_{x \rightarrow 1^-} f(x) = f(1)$$

$$\Rightarrow \lim_{h \rightarrow 0} f(1 - h) = \lim_{h \rightarrow 0} f(1 + h) = f(1)$$

using $f(x + y) = f(x) f(y)$, we get

$$\Rightarrow \lim_{h \rightarrow 0} f(1) f(-h) = \lim_{h \rightarrow 0} f(1) f(h) = f(1)$$

$$\Rightarrow \lim_{h \rightarrow 0} f(-h) = \lim_{h \rightarrow 0} f(h) = 1 \quad \dots\dots\dots(i)$$

Now consider some arbitrary point $x = a$

$$\text{Left hand limit} = \lim_{h \rightarrow 0} f(a - h) = \lim_{h \rightarrow 0} f(a) f(-h)$$

$$= f(a) \lim_{h \rightarrow 0} f(-h) = f(a) \quad \dots\dots\dots \text{using (i)}$$

$$\text{Right hand limit} = \lim_{h \rightarrow 0} f(a + h) = \lim_{h \rightarrow 0} f(a) f(h)$$

$$= f(a) \lim_{h \rightarrow 0} f(h) = f(a) \quad \dots\dots\dots \text{using (i)}$$

Hence at any arbitrary point ($x = a$)

$$\text{L.H.L.} = \text{R.H.L.} = f(a)$$

\Rightarrow function is continuous for all values of x .

Example : 15

$$f(x) = \begin{cases} 1+x & ; 0 \leq x \leq 2 \\ 3-x & ; 2 < x \leq 3 \end{cases}$$

Determine the form of $g(x) = f(f(x))$ and hence find the point of discontinuity of g , if any

Solution

$$g(x) = f(f(x)) = \begin{cases} 1+x & ; 0 \leq x \leq 2 \\ 3-x & ; 2 < x \leq 3 \end{cases} = \begin{cases} f(1+x) & ; 0 \leq x \leq 1 \\ f(1+x) & ; 1 < x \leq 2 \\ f(3-x) & ; 2 < x \leq 3 \end{cases}$$

Now $x \in [0, 1] \Rightarrow (1+x) \in [1, 2]$
 $x \in (1, 2] \Rightarrow (1+x) \in (2, 3]$
 $x \in (2, 3] \Rightarrow (3-x) \in [0, 1]$

Hence

$$g(x) = \begin{cases} f(1+x) & \text{for } 0 \leq x \leq 1 \Rightarrow 1 \leq x+1 \leq 2 \\ f(1+x) & \text{for } 1 < x \leq 2 \Rightarrow 2 < x+1 \leq 3 \\ f(3-x) & \text{for } 2 < x \leq 3 \Rightarrow 0 \leq 3-x < 1 \end{cases}$$

Now if $(1+x) \in [1, 2]$, then $f(1+x) = 1 + (1+x) = 2+x$ (i)
 [from the original definition of $f(x)$]

Similarly if $(1+x) \in (2, 3]$, then
 $f(1+x) = 3 - (1+x) = 2-x$ (ii)

If $(3-x) \in (0, 1)$, then
 $f(3-x) = 1 + (3-x) = 4-x$ (iii)

Using (i), (ii) and (iii), we get $g(x) = \begin{cases} 2+x & ; 0 \leq x \leq 1 \\ 2-x & ; 1 < x \leq 2 \\ 4-x & ; 2 < x \leq 3 \end{cases}$

Now we will check the continuity of $g(x)$ at $x = 1, 2$

At $x = 1$

L.H.L. = $\lim_{x \rightarrow 1^-} g(x) = \lim_{x \rightarrow 1^-} (2+x) = 3$

R.H.L. = $\lim_{x \rightarrow 1^+} g(x) = \lim_{x \rightarrow 1^+} (2-x) = 1$

As L.H.L., $g(x)$ is discontinuous at $x = 1$

At $x = 2$

L.H.L. = $\lim_{x \rightarrow 2^-} g(x) = \lim_{x \rightarrow 2^-} (2-x) = 0$

R.H.L. = $\lim_{x \rightarrow 2^+} g(x) = \lim_{x \rightarrow 2^+} (4-x) = 2$

As L.H.L. \neq R.H.L., $g(x)$ is discontinuous at $x = 2$

Example : 16

Discuss the continuity of $f(x) = \begin{cases} e^{1/x} - 1 & ; x \neq 0 \\ 0 & ; x = 0 \end{cases}$ at the point $x = 0$

Solution

LHL = $\lim_{x \rightarrow 0^-} \frac{e^{\frac{1}{x}} - 1}{\frac{1}{x} + 1} = \lim_{t \rightarrow -\infty} \frac{e^t - 1}{e^t + 1} = \frac{0-1}{0+1} = -1$

$$\text{RHL} = \lim_{x \rightarrow 0^+} \frac{e^{\frac{1}{x}} - 1}{\frac{1}{e^x} + 1} = \lim_{t \rightarrow \infty} \frac{e^t - 1}{e^t + 1} = \lim_{t \rightarrow \infty} \frac{1 - e^{-t}}{1 + e^{-t}}$$

$$\Rightarrow \text{R.H.L.} = \frac{1-0}{1+0} = 1$$

$$\Rightarrow \text{L.H.L.} \neq \text{R.H.L.} \Rightarrow f(x) \text{ is discontinuous at } x = 0$$

Example : 17

Discuss the continuity of the function $g(x) = [x] + [-x]$ at integral values of x .

Solution

Let us simplify the definition of the function

(i) If x is an integer :

$$[x] = x \text{ and } [-x] = -x \Rightarrow g(x) = x - x = 0$$

(ii) If x is not integer :

let $x = n + f$ where n is an integer and $f \in (0, 1)$

$$\Rightarrow [x] = [n + f] = n$$

$$\text{and } [-x] = [-n - f] = [(-n - 1) + (1 - f)] = -n - 1$$

(because $0 < f < 1 \Rightarrow 0, (1 - f) < 1$)

$$\text{Hence } g(x) = [x] + [-x] = n + (-n - 1) = -1$$

$$\text{So we get : } g(x) = \begin{cases} 0 & , \text{ if } x \text{ is an integer} \\ -1 & , \text{ if } x \text{ is not an integer} \end{cases}$$

Let us discuss the continuity of $g(x)$ at a point $x = a$ where $a \in I$

$$\text{L.H.L.} = \lim_{x \rightarrow a^-} g(x) = -1$$

\because as $x \rightarrow a^-$, x is not an integer

$$\text{R.H.L.} = \lim_{x \rightarrow a^+} g(x) = -1$$

as $x \rightarrow a^+$, x is not an integer

but $g(a) = 0$ because a is an integer

Hence $g(x)$ has a removable discontinuity at integral values of x .

Example : 18

Which of the following functions are even/odd?

(a) $f(x) = \frac{a^x - 1}{a^x + 1}$

(b) $f(x) = x \log \left(\frac{1+x}{1-x} \right)$

(c) $f(x) = |x|$

(d) $f(x) = \log \left(x + \sqrt{x^2 + 1} \right)$

Solution

(a) $f(-x) = \frac{a^{-x} - 1}{a^{-x} + 1} = \frac{1 - a^x}{1 + a^x} = -f(x) \Rightarrow f(x) \text{ is odd}$

(b) $f(-x) = -x \log \left(\frac{1-x}{1+x} \right) = x \log \left(\frac{1-x}{1+x} \right)^{-1} = x \log \left(\frac{1+x}{1-x} \right) = f(x)$

$\Rightarrow f(x)$ is even

(c) $f(-x) = |-x| = |x| = f(x)$

$\Rightarrow f(x)$ is even

Let the base of this exponential function be a.

Thus $f(x) = a^x$

It is given that $f(1) = 2$. So we can make

$f(1) = a^1 = 2 \Rightarrow a = 2$

Hence, the function is $f(x) = 2^x$ (i)

[Alternatively, we have

$f(x) = f(x - 1 + 1) = f(x - 1) f(1) = f(x - 2 + 1) f(1) = f(x - 2) [f(1)]^2 = \dots = [f(1)]^x = 2^x]$

Using equation (i), the given expression reduces to :

$$\sum_{k=1}^n 2^{a+k} = 16 (2^n - 1)$$

$$\Rightarrow \sum_{k=1}^n 2^a \cdot 2^k = 16 (2^n - 1)$$

$$\Rightarrow 2^a \sum_{k=1}^n 2^k = 16 (2^n - 1)$$

$$\Rightarrow 2^a (2 + 4 + 8 + 16 + \dots + 2^n) = 16 (2^n - 1)$$

$$\Rightarrow 2^a \left[\frac{2(2^n - 1)}{2 - 1} \right] = 16 (2^n - 1)$$

$$\Rightarrow 2^{a+1} = 16 \Rightarrow 2^{a+1} = 2^4$$

$$\Rightarrow a + 1 = 4 \Rightarrow a = 3$$

Example : 21

Evaluate the following limits :

(i) $\lim_{x \rightarrow 2} \frac{3^x + 3^{3-x} - 12}{3^{3-x} - 3^{x/2}}$ (ii) $\lim_{x \rightarrow \pi/3} \frac{\tan^3 x - 3 \tan x}{\cos\left(x + \frac{\pi}{6}\right)}$ (iii) $\lim_{x \rightarrow -\infty} \frac{x^4 \sin \frac{1}{x} + x^2}{1 + |x^3|}$

Solution

(i) Let $L = \lim_{x \rightarrow 2} \frac{3^x + 3^{3-x} - 12}{3^{3-x} - 3^{x/2}}$

$$\Rightarrow L = \lim_{x \rightarrow 2} \frac{3^x + \frac{27}{3^x} - 12}{\frac{27}{3^x} - 3^{x/2}}$$

$$\Rightarrow L = \lim_{x \rightarrow 2} \frac{3^{2x} - 12 \cdot 3^x + 27}{(3^{x/2})^3 - 3^3}$$

$$\Rightarrow L = \lim_{x \rightarrow 2} \frac{(3^x - 9)(3^x - 3)}{(3^{x/2} - 3)(3^x + 9 + 3 \cdot 3^{x/2})}$$

$$\Rightarrow L = \lim_{x \rightarrow 2} \frac{(3^{x/2} + 3)(3^x - 3)}{(3^x + 3 \cdot 3^{x/2} + 9)}$$

$$\Rightarrow L = \frac{6 \cdot 6}{9 + 3 \cdot 3 + 9} = \frac{36}{27} = \frac{4}{3}$$

$$(ii) \quad \text{Let } L = L = \lim_{x \rightarrow \pi/3} \frac{\tan^3 x - 3 \tan x}{\cos\left(x + \frac{\pi}{6}\right)} \quad \text{and} \quad x - \frac{\pi}{3} = t$$

$$\Rightarrow L = \lim_{t \rightarrow 0} \frac{\tan^3\left(1 + \frac{\pi}{3}\right) - 3 \tan\left(t + \frac{\pi}{3}\right)}{\cos\left(t + \frac{\pi}{2}\right)}$$

$$\Rightarrow L = \lim_{t \rightarrow 0} \frac{\tan(3t + \pi) \left[3 \tan^2\left(1 + \frac{\pi}{3}\right) - 1 \right]}{-\sin t}$$

$$\Rightarrow L = \lim_{t \rightarrow 0} \frac{-\tan(3t)}{-\sin t} \cdot \lim_{t \rightarrow 0} \left[3 \tan^2\left(t + \frac{\pi}{3}\right) - 1 \right]$$

$$\Rightarrow L = 3 \lim_{t \rightarrow 0} \frac{\tan(3t)}{3t} \times \lim_{t \rightarrow 0} \frac{1}{\sin t} \times \lim_{t \rightarrow 0} \left[3 \tan^2\left(1 + \frac{\pi}{3}\right) - 1 \right]$$

$$\Rightarrow L = 3 \times 1 \times 1 \times 8 = 24$$

$$(iii) \quad \text{Let } L = \lim_{x \rightarrow -\infty} \frac{x^4 \sin \frac{1}{x} + x^2}{1 + |x^3|}$$

Divided Numerator and Denominator by x^3 to get

$$L = \lim_{x \rightarrow -\infty} \frac{x \sin \frac{1}{x} + x^2}{\frac{1}{x^3} + |x|^3} = \frac{\frac{\sin \frac{1}{x}}{1/x} + \frac{1}{x}}{\frac{1}{x^3} + \frac{(-x)^3}{x^3}} \quad (\because \text{for } x < 0, |x^3| = -x^3)$$

$$\Rightarrow L = \lim_{x \rightarrow -\infty} \frac{\rightarrow (1)^+ \rightarrow (0)}{\rightarrow (0)^+ \rightarrow (-1)} = -1$$

Example : 22

$$\text{Let } f(x) = \begin{cases} (1 + |\sin x|)^{a/|\sin x|} & ; \quad \frac{\pi}{6} < x < 0 \\ b & ; \quad x = 0 \\ \frac{\tan 2x}{e^{\tan 3x}} & ; \quad 0 < x < \frac{\pi}{6} \end{cases}$$

Determine a and b such that $f(x)$ is continuous at $x = 0$

Solution

Left hand limit at $x = 0$

$$\text{L.H.L.} = \lim_{x \rightarrow 0^-} f(x) = \lim_{x \rightarrow 0^-} \left[(1 + |\sin x|)^{\frac{a}{|\sin x|}} \right]$$

$$\Rightarrow \text{L.H.L.} = \lim_{h \rightarrow 0} f(0 - h)$$

$$\Rightarrow \text{L.H.L.} = \lim_{h \rightarrow 0} \left[(1 + |\sinh|)^{\frac{a}{|\sinh|}} \right] = e^x \quad \left[\text{using : } \lim_{t \rightarrow 0} (1+t)^{\frac{1}{t}} = e \right]$$

Right hand limit $x = 0$

$$\text{R.H.L.} = \lim_{x \rightarrow 0^+} f(x) = \lim_{x \rightarrow 0^+} e^{\frac{\tan 2x}{\tan 3x}}$$

$$\Rightarrow \text{R.H.L.} = \lim_{h \rightarrow 0} f(0 + h)$$

$$\Rightarrow \text{R.H.L.} = \lim_{h \rightarrow 0} e^{\frac{\tan 2h}{\tan 3h}}$$

$$\Rightarrow \text{R.H.L.} = \lim_{h \rightarrow 0} e^{\frac{2}{3} \left(\frac{\tan 2h}{2h} \cdot \frac{3h}{\tan 3h} \right)} = e^{\frac{2}{3}}$$

for continuity

$$\text{L.H.L.} = \text{R.H.L.} = f(0)$$

$$\Rightarrow e^x = e^{\frac{2}{3}} = b$$

$$\Rightarrow a = \frac{2}{3}, b = e^{2/3}$$

Example : 23

Discuss the continuity of $f(x)$ in $[0, 2]$ where $f(x) = \lim_{n \rightarrow \infty} \left(\sin \frac{\pi x}{2} \right)^{2n}$

Solution

$$\text{Since } \lim_{n \rightarrow \infty} x^{2n} = \begin{cases} 0 & ; |x| < 1 \\ 1 & ; |x| = 1 \end{cases}$$

$$\therefore f(x) = \lim_{n \rightarrow \infty} \left(\sin \frac{\pi x}{2} \right)^{2n}$$

$$= \begin{cases} 0 & ; \left| \sin \frac{\pi x}{2} \right| < 1 \\ 1 & ; \left| \sin \frac{\pi x}{2} \right| = 1 \end{cases}$$

Thus $f(x)$ is continuous for all x , except for those values of x for which $\left| \sin \frac{\pi x}{2} \right| = 1$

i.e. x is an odd integer

$$\Rightarrow x = (2n + 1) \quad \text{where } x \in \mathbb{I}$$

Check continuity at $x = (2n + 1)$:

$$\text{L.H.L.} = \lim_{x \rightarrow 2n+1} f(x) = 0 \quad \dots\dots\dots(i)$$

$$\text{and } f(2n + 1) = 1 \quad \dots\dots\dots(ii)$$

from (i) and (ii), we get :

$$\text{L.H.L.} \neq f(2n + 1),$$

$$\Rightarrow f(x) \text{ is discontinuous at } x = 2n + 1$$

(i.e. at odd integers)

Hence $f(x)$ is discontinuous at $x = (2n + 1)$.

Example : 24

$$\text{Let } f(x) = \begin{cases} \frac{1 - \cos 4x}{x^2} & ; x < 0 \\ a & ; x = 0 \\ \frac{\sqrt{x}}{\sqrt{16 + \sqrt{x}} - 4} & ; x > 0 \end{cases}$$

Determine the value of a, if possible, so that the function is continuous at $x = 0$.

Solution

$$\text{It is given that } f(x) = \begin{cases} \frac{1 - \cos 4x}{x^2} & ; x < 0 \\ a & ; x = 0 \\ \frac{\sqrt{x}}{\sqrt{16 + \sqrt{x}} - 4} & ; x > 0 \end{cases}$$

is continuous at $x = 0$. So we can take :

$$\lim_{x \rightarrow 0^-} f(x) = f(0) = \lim_{x \rightarrow 0^+} f(x)$$

Left hand limit at $x = 0$

$$\text{L.H.L.} = \lim_{t \rightarrow 0^-} f(x) = \lim_{t \rightarrow 0^-} \frac{1 - \cos 4x}{x^2}$$

$$\text{Now, L.H.L.} = \lim_{h \rightarrow 0} f(0 - h)$$

$$\Rightarrow \text{L.H.L.} = \lim_{h \rightarrow 0} \frac{1 - \cos 4h}{h^2} = \lim_{h \rightarrow 0} \frac{2 \sin^2 2h}{h^2} = 8 \quad \left[\text{using : } \lim_{t \rightarrow 0} \frac{\sin t}{t} = 1 \right]$$

Right hand limit at $x = 0$

$$\text{R.H.L.} = \lim_{t \rightarrow 0^+} f(x) = \lim_{t \rightarrow 0^+} \frac{\sqrt{x}}{\sqrt{16 + \sqrt{x}} - 4}$$

$$\text{Now, R.H.L.} = \lim_{h \rightarrow 0} f(0 + h)$$

$$\Rightarrow \text{R.H.L.} = \lim_{h \rightarrow 0} \frac{\sqrt{h}}{\sqrt{16 + \sqrt{h}} - 4}$$

Rationalising denominator to get :

$$\Rightarrow \text{R.H.L.} = \lim_{h \rightarrow 0} \frac{\sqrt{h}}{\sqrt{h}} \left(\sqrt{16 + \sqrt{h}} + 4 \right) = 8$$

For function $f(x)$ to be continuous at $x = 0$,

$$\text{L.H.L.} = \text{R.H.L.} = f(0)$$

$$\Rightarrow 8 = 8 = a$$

$$\Rightarrow a = 8$$

Example : 25

Let $\{x\}$ and $[x]$ denote the fractional and integral part of a real number x respectively.

Solve $4\{x\} = x + [x]$.

Solution

We can write x as :

$x = \text{integral part} + \text{fractional part}$

$$\Rightarrow x = [x] + \{x\}$$

The given equation is $4\{x\} = x + [x]$

$$\begin{aligned}
\Rightarrow 4\{x\} &= [x] + \{x\} + [x] \\
\Rightarrow 3\{x\} &= 2[x] \quad \dots\dots\dots(i) \\
\Rightarrow 3\{x\} &\text{ is an even integer} \\
\text{But we have } 0 \leq \{x\} &< 1 \\
\Rightarrow 0 \leq 3\{x\} &< 3 \\
\Rightarrow 3\{x\} &= 0, 2 \quad [\text{become } 3\{x\} \text{ is even}] \\
\Rightarrow \{x\} = 0, \frac{2}{3} &\quad \text{and} \quad [x] = \frac{3}{2} \quad \{x\} = 0, 1 \quad \dots\dots\dots \text{(using i)} \\
\Rightarrow x = 0 + 0 &\quad \text{or} \quad x = \frac{2}{3} + 1 \\
\Rightarrow x = 0 &\quad \text{or} \quad x = \frac{5}{3}
\end{aligned}$$

Example : 26

Discuss the continuity of $f(x)$ in $[0, 2]$ where $f(x) = \begin{cases} [\cos \pi x] & ; x \leq 1 \\ |2x - 3| [x - 2] & ; x > 1 \end{cases}$

where $[]$: represents the greatest integer function.

Solution

First of all find critical points where $f(x)$ may be discontinuous

Consider $x \in [0, 1]$:

$$f(x) = [\cos \pi x]$$

$[f(x)]$ is discontinuous where $f(x) \in I$

$$\Rightarrow \cos \pi x = I$$

In $[0, 1]$, $\cos \pi x$ is an integer at $x = 0$, $x = \frac{1}{2}$ and $x = 1$

$$\Rightarrow x = 0, x = \frac{1}{2} \text{ and } x = 1 \text{ are critical points} \quad \dots\dots\dots(i)$$

Consider $x \in (1, 2]$:

$$\begin{aligned}
f(x) &= [x - 2] |2x - 3| \\
\text{In } x \in (1, 2) \quad [x - 2] &= -1 \quad \text{and} \\
\text{for } x = 2 \quad ; \quad [x - 2] &= 0
\end{aligned}$$

$$\text{Also } |2x - 3| = 0 \quad \Rightarrow \quad x = \frac{3}{2}$$

$$\Rightarrow x = \frac{3}{2} \text{ and } x = 2 \text{ are critical points} \quad \dots\dots\dots(ii)$$

combining (i) and (ii), critical points are $0, \frac{1}{2}, 1, \frac{3}{2}, 2$

On dividing $f(x)$ about the 5 critical points, we get

$$f(x) = \begin{cases} 1 & ; x = 0 & \because \cos(\pi 0) = 1 \\ 0 & ; 0 < x \leq \frac{1}{2} & \because 0 \leq \cos \pi x < 1 \Rightarrow [\cos \pi x] = 0 \\ -1 & ; \frac{1}{2} < x \leq 1 & \because -1 \leq \cos \pi x < 0 \Rightarrow [\cos \pi x] = -1 \\ -1(3 - 2x) & ; 1 < x \leq \frac{3}{2} & \because |2x - 3| = 3 - 2x \text{ and } [x - 2] = -1 \\ -1(2x - 3) & ; \frac{3}{2} < x < 2 & \because |2x - 3| = 2x - 3 \text{ and } [x - 2] = -1 \\ 0 & ; 2 & \because [x - 2] = 0 \end{cases}$$

Checking continuity at $x = 0$:

$$\text{R.H.L.} = \lim_{x \rightarrow 0^+} (0) = 0 \quad \text{and} \quad f(0) = 1$$

As, $\text{R.H.L.} \neq f(x)$

$\Rightarrow f(x)$ is discontinuous at $x = 0$

Checking continuity at $a + x = 1/2$

$$\text{L.H.L.} = \lim_{x \rightarrow \frac{1}{2}^-} f(x) = 0$$

$$\text{R.H.L.} = \lim_{x \rightarrow \frac{1}{2}^+} f(x) = -1$$

As $\text{L.H.L.} \neq \text{R.H.L.}$,

$f(x)$ is discontinuous at $x = \frac{1}{2}$.

Checking continuity at $x = 1$:

$$\text{L.H.L.} = \lim_{x \rightarrow 1^-} f(x) = -1$$

$$\text{R.H.L.} = \lim_{x \rightarrow 1^+} f(x) = -1 = \lim_{x \rightarrow 1^+} (2x - 3) = -1$$

and $f(1) = -1$

As $\text{L.H.L.} = \text{R.H.L.} = f(1)$

$f(x)$ is continuous at $x = 1$

Checking continuity at $x = 3/2$:

$$\text{L.H.L.} = \lim_{x \rightarrow \frac{3}{2}^-} (2x - 3) = 0$$

$$\text{R.H.L.} = \lim_{x \rightarrow \frac{3}{2}^+} (3 - 2x) = 0 \quad \text{and} \quad f\left(\frac{3}{2}\right) = 0$$

As $\text{L.H.L.} = \text{R.H.L.} = f\left(\frac{3}{2}\right)$,

$f(x)$ is continuous at $x = 3/2$

Checking continuity at $x = 2$:

$$\text{L.H.L.} = \lim_{x \rightarrow 2^-} (3 - 2x) = -1 \quad \text{and} \quad f(2) = 0$$

As $\text{L.H.L.} \neq f(2)$,

$f(x)$ is discontinuous at $x = 2$

Example : 27

If $f(x) = \frac{\sin 2x + A \sin x + B \cos x}{x^3}$ is continuous at $x = 0$, find the values of A and B. Also find $f(0)$.

Solution

As $f(x)$ is continuous at $x = 0$

$f(0) = \lim_{x \rightarrow a} f(x)$ and both $f(0)$ and $\lim_{x \rightarrow a} f(x)$ are finite

$$\Rightarrow f(0) = \lim_{x \rightarrow 0} \frac{\sin 2x + A \sin x + B \cos x}{x^3}$$

As denominator $\rightarrow 0$ as $x \rightarrow 0$,

\therefore Numerator should also $\rightarrow 0$ as $x \rightarrow 0$.

Which is possible only if (for $f(0)$ to be finite)

$$\sin 2(0) + A \sin (0) + B \cos 0 = 0$$

$$\Rightarrow B = 0$$

$$\therefore f(0) = \lim_{x \rightarrow 0} \frac{\sin 2x + A \sin x}{x^2}$$

$$\Rightarrow f(0) = \lim_{x \rightarrow 0} \left(\frac{\sin x}{x} \right) \left(\frac{2 \cos x + A}{x^2} \right) = \lim_{x \rightarrow 0} \left(\frac{2 \cos x + A}{x^2} \right)$$

Again we can see that Denominator $\rightarrow 0$ as $x \rightarrow 0$

\therefore Numerator should also approach 0 as $x \rightarrow 0$ (for $f(0)$ to be finite)

$$\Rightarrow 2 + A = 0 \quad \Rightarrow \quad A = -2$$

$$\Rightarrow f(0) = \lim_{x \rightarrow 0} \left(\frac{2 \cos x - 2}{x^2} \right) = \lim_{x \rightarrow 0} \left(\frac{-4 \sin^2 \frac{x}{2}}{x^2} \right) = \lim_{x \rightarrow 0} \left(\frac{-\sin^2 \frac{x}{2}}{\frac{x^2}{4}} \right) = -1$$

So we get $A = -2$, $B = 0$ and $f(0) = -1$

Example : 28

Evaluate $\lim_{x \rightarrow 0} \frac{64^x - 32^x - 16^x + 4^x + 2^x - 1}{(\sqrt{3 + \cos x} - 2) \sin x}$

Solution

$$\text{Let } L = \lim_{x \rightarrow 0} \frac{64^x - 32^x - 16^x + 4^x + 2^x - 1}{(\sqrt{3 + \cos x} - 2) \sin x}$$

On rationalising the denominator, we get

$$L = \lim_{x \rightarrow 0} \frac{2^{6x} - 2^{5x} - 2^{4x} + 2^{2x} + 2^x - 1}{(\cos x - 1) \sin x} (\sqrt{3 + \cos x} + 2)$$

On factorising the numerator, we get

$$L = \lim_{x \rightarrow 0} \frac{(2^x - 1)[2^{5x} - 2^{5x}(2^x - 1) + 1]}{(\cos x - 1) \sin x} \times \lim_{x \rightarrow 0} (\sqrt{3 + \cos x} + 2)$$

$$\Rightarrow L = \lim_{x \rightarrow 0} \frac{(2^x - 1)[(2^{5x} - 2^{3x}) - (2^{2x} - 1)]}{(\cos x - 1) \sin x} \times 4$$

$$\Rightarrow L = \lim_{x \rightarrow 0} \frac{(2^x - 1)(2^{2x} - 1)(2^{3x} - 1)}{(\cos x - 1) \sin x} \times 4$$

$$\Rightarrow L = 4 \lim_{x \rightarrow 0} \left(\frac{2^x - 1}{x} \right) \times 2 \lim_{x \rightarrow 0} \left(\frac{2^{2x} - 1}{2x} \right) \times 3$$

$$\lim_{x \rightarrow 0} \left(\frac{2^{3x} - 1}{3x} \right) \times \lim_{x \rightarrow 0} \left(\frac{x^2}{-2 \sin^2(x/2)} \right) \times \lim_{x \rightarrow 0} \left(\frac{x}{\sin x} \right)$$

$$\Rightarrow L = 4 (\ln 2) = 2 (\ln 2) \times 3 (\ln 2) \times (-2) \Rightarrow L = -48 (\ln 2)^3$$

Example : 29

(i) If f is an even function defined on the interval $(-5, 5)$, then find the four real values of x satisfying

$$\text{the equation } f(x) = f\left(\frac{x+1}{x+2}\right).$$

(ii) Evaluate : $\lim_{x \rightarrow 0} \left(\frac{1+5x^2}{1+3x^2} \right)^{\frac{1}{x^2}}$.

- (iii) If $f(x) = \sin^2 x + \sin^2 \left(x + \frac{\pi}{3} \right) = \cos x \cos \left(x + \frac{\pi}{3} \right)$ and $g \left(\frac{5}{4} \right) = 1$, then find $g[f(x)]$.
- (iv) Let $f(x) = [x] \sin \frac{(\pi)}{[x+1]}$ where $[]$ denotes the greater integer function. Find the domain of $f(x)$ and the points of discontinuity of $f(x)$ in the domain.

Solution

(i) It is given that $f(x) = f \left(\frac{x+1}{x+2} \right)$

$$\Rightarrow x = \left(\frac{x+1}{x+2} \right) \Rightarrow x^2 + x - 1 = 0$$

$$\Rightarrow x = \frac{-1 \pm \sqrt{5}}{2} \quad \dots\dots\dots(i)$$

As $f(x)$ is even, $f(x) = f(-x)$

$$-x = \left(\frac{x+1}{x+2} \right) \Rightarrow x^2 + 3x + 1 = 0 \Rightarrow x = \frac{-3 \pm \sqrt{5}}{2} \quad \dots\dots\dots(ii)$$

One combining (i) and (ii), we get :

$$x = \frac{-1 \pm \sqrt{5}}{2} \text{ and } x = \frac{-3 \pm \sqrt{5}}{2} .$$

(ii) Let $L = \lim_{x \rightarrow 0} \left(\frac{1+5x^2}{1+3x^2} \right)^{\frac{1}{x^2}}$

$$\Rightarrow L = \lim_{x \rightarrow 0} \left(1 + \frac{1+5x^2}{1+3x^2} - 1 \right)^{\frac{1}{x^2}}$$

$$\Rightarrow L = \lim_{x \rightarrow 0} \left(1 + \frac{2x^2}{1+3x^2} \right)^{\frac{1}{x^2}}$$

$$\Rightarrow L = \lim_{x \rightarrow 0} e^{\left(\frac{2}{1+3x^2} \right)} = e^2 \quad \left[\text{using : } \lim_{t \rightarrow 0} (1+t)^{\frac{1}{t}} = e \right]$$

(iii) It is given that $f(x) = 1 - \cos^2 x + \sin^2 \left(x + \frac{\pi}{3} \right) + \cos x \cos \left(x + \frac{\pi}{3} \right)$

$$= 1 - \left[\cos^2 x - \sin^2 \left(x + \frac{\pi}{3} \right) \right] + \frac{1}{2} \left[2 \cos x \cos \left(x + \frac{\pi}{3} \right) \right]$$

$$= 1 - \cos \left(2x + \frac{\pi}{3} \right) \cos \frac{\pi}{3} + \frac{\cos \left(2x + \frac{\pi}{3} \right)}{2} + \frac{\cos \left(\frac{\pi}{3} \right)}{2} = 1 + \frac{\cos \left(\frac{\pi}{3} \right)}{2} = \frac{5}{4}$$

⇒ For all values of x, $f(x) = \frac{5}{4}$. (constant function)

Hence, $g[f(x)] = g\left(\frac{5}{4}\right)$

But $g\left(\frac{5}{4}\right) = 1 \Rightarrow g[f(x)] = 1$

Hence, $g[f(x)] = 1$ for all values of x

(iv) Let $f(x) = [x] \sin \frac{(\pi)}{[x+1]}$

Domain of f(x) is $x \in \mathbb{R}$ excluding the point where $[x + 1] = 0$
(∵ denominator cannot be zero)

Find values of x which satisfy $[x + 1] = 0$

$[x + 1] = 0$

⇒ $0 \leq x + 1 < 1$

⇒ $-1 \leq x < 0$

i.e. for all $x \in [-1, 0)$, denominator is zero.

So, domain is $x \in \mathbb{R} [-1, 0)$

⇒ Domain is $x \in (-\infty, -1) \cup [0, \infty)$

Point of Discontinuity

As greatest integer function is discontinuous at integer points, f(x) is continuous for all non-integer points.

Checking continuity at $x = a$ (where $a - 1$)

L.H.L. = $\lim_{h \rightarrow 0} [a - h] \sin \left(\frac{\pi}{[a + 1 - h]} \right)$

⇒ L.H.L. = $(a - 1) \sin \left(\frac{\pi}{a} \right)$ (i)

R.H.L. = $\lim_{h \rightarrow 0} [a + h] \sin \left(\frac{\pi}{[a + 1 + h]} \right)$

⇒ L.H.L. = $a \sin \left(\frac{\pi}{a + 1} \right)$ (ii)

From (i) and (ii), L.H.L. \neq R.H.L.

⇒ f(x) is discontinuous at $x = a$ (i.e. at integer values of x)

So, points of discontinuity are $x \in I \cap D$. (i.e. integers lying in the set of domain)

⇒ $x \in I - \{-1\}$.