Example : 1

How many (a) 5 – digit (b) 3–digit numbers can be formed using 1, 2, 3, 7, 9 without any repetition of digits?

Solution

(a) 5-digit numbers

Making a 5-digit number is equivalent to filling 5 places

Places :	☐	☐	☐	☐	☐
No. of choices :	**1**	**2**	**3**	**4**	**5**

The last place (unit's place) can be filled in 5 ways using any of the five given digits.

The ten's place can be filled in four ways using any of the remaining 4 digits.

The number of choices for other places can be calculated in the same way.

No. of ways to fill all five places = $5 \times 4 \times 3 \times 2 \times 1 = 5! = 120$

$\Rightarrow$ 120 five-digit numbers can be formed

(b) 3-digit numbers

Making a three-digit number is equivalent to filling three places (unit's, ten's, hundred's)

Places :	☐	☐	☐
No. of choices :	**3**	**4**	**5**

No. of ways to fill all the three places = $5 \times 4 \times 3 = 60$

$\Rightarrow$ 60 three-digit numbers can be formed

Example : 2

How many 3-letter words can be formed using a, b, c, d, e if :

(a) repetition is not allowed (b) repetition is allowed ?

Solution

(a) Repetition is not allowed :

The number of words that can be formed is equal to the number of ways to fill the three places

Places :	☐	☐	☐
No. of choices :	**5**	**4**	**3**

$\Rightarrow$ $5 \times 4 \times 3 = 60$ words can be formed

(b) Repetition is allowed:

The number of words that can be formed is equal to the number of ways to fill the three places.

Places :	☐	☐	☐
No. of choices :	**5**	**5**	**5**

First place can be filled in five ways (a, b, c, d, e)

If repetition is allowed, all the remaining places can be filled in five ways using a, b, c, d, e.

No. of words = $5 \times 5 \times 5 = 125$ words can be formed

Example : 3

How many four-digit numbers can be formed using the digits 0, 1, 2, 3, 4, 5?

Solution

For a four-digit number, we have to fill four places and - cannot appear in the first place (thousand's place)

Places :	☐	☐	☐	☐
No. of choices :	**5**	**5**	**4**	**3**

For the first place, there are five choices (1, 2, 3, 4, 5); Second place can then be filled in five ways (0 and remaining four-digits); Third place can be filled in four ways (remaining four-digits); Fourth place can be filled in three ways (remaining three-digit).

Total number of ways = $5 \times 5 \times 4 \times 3 = 300$

$\Rightarrow$ 300 four-digits numbers can be formed

Example : 4

In how many ways can six persons be arranged in a row?

Solution

Arranging a given set of n different objects is equivalent to filling n places

So arranging six persons along a row is equivalent to filling 6 places

Places :						
No. of choices :	**6**	**5**	**4**	**3**	**2**	**1**

No. of ways to fill all places = 6 × 5 × 4 × 3 × 2 × 1 = 6! = 720

Hence 720 arrangements are possible

Example : 5

How many nin-letter words can be formed by using the letters of the words

(a) E Q U A T I O N S (b) A L L A H A B A D

Solution

(a) All nine letters in the word EQUATIONS are different

Hence number of words = ${}^{9}P_{9}$ = 9! = 362880

(b) ALLAHABAD contains LL, AAAA, H, B, D.

No. of words = $\frac{9!}{2!\,4!} = \frac{9\times8\times7\times6\times5}{2}$ = 7560

Example : 6

(a) How many words can be made by using the letters of the word C O M B I N E all at a time?

(b) How many of these words begin and end with a vowel?

(c) In how many of these words do the vowels and the consonants occupy the same relative positions as in C O M B I N E ?

Solution

(a) The total number of words – arrangement of seven letters taken all at a time = ${}^{7}P_{7}$ = 7! = 5040

(b) The corresponding choices for all the places are as follows :

Places	**vowel**						**vowel**
No. of choices	**3**	**5**	**4**	**3**	**2**	**1**	**2**

As there are three vowels (O I E), first place can be filled in three ways and the last place can be filled in two ways. The rest of the places can be filled in 5 ! ways using five remaining letters.

No. of words = 3 × 5! × 2 = 720

(c) Vowels should be at second, fifth and seventh positions

They can be arranged in 3! ways

Consonants should be at first, third, fourth and sixth positions.

They can be arranged here in 4! ways

Total number of words = 3! × 4! = 144

Example : 7

How many words can be formed using the letters of the word TRIANGLE so that

(a) A and N are always together?

(b) T, R, I are always together?

Solution

(a) Assume (AN) as a single letter. Now there are seven letters in all : (AN), T, R, I, G, L, E

Seven letters can be arranged in 7! ways

All these 7! words will contain A and N together. A and N can now be arranged among themselves in 2! ways (AN and NA).

Hence total number of words = 7! 2! = 10080

(b) Assume (TRI) as a single letters

(i) The letters : (TRI), A, N, G, L, E can be arranged in 6! ways

(ii) TRI can be arranged among themselves in 3! ways

Total number of words = 6! 3! = 4320

Example : 8

There are 9 candidates for an examination out of which 3 are appearing in Mathematices and remaining 6 are appearing in different subjects. In how many ways can they be seated in a row so that no two Mathematices candidates are together?

Solution

Divide the work in two operations.

(i) First, arrange the remaining candidate in 6! ways

(ii) Place the three Mathematices candidate in the row of six other candidate so that no two of them are together.

X : Places available for Mathematices candidates.

O : Others

X	O	X	O	X	O	X	O	X	O	X	O	X

In any arrangement of 6 other candidates (O), there are seven places available for Mathematices candidates so that they are not together. Now 3 Mathematices candidates can be placed in these 7 places in $^7P_3$ ways.

Hence total number of arrangements = $6!\ {}^7P_3 = 720 \times \frac{7!}{4!} = 151200$

Example : 9

(a) How many triangle can be formed by joining the vertices of a bexagon?

(b) How many diagonals are there in a polygon with n sides?

Solution

(a) Let $A_1, A_2, A_3, \ldots\ldots, A_6$ be the vertices of the bexagon. One triangle is formed by selecting a group of 3 points from 6 given vertices.

No. of triangles = No. of groups of 3 each from 6 points = $^6C_3 = \frac{6!}{3!\,3!} = 20$

(b) No. of lines that can be formed by using the given vertices of a polygon = No. of groups of 2 points each selected from the n points

$= {}^nC_2 = \frac{n!}{n!(n-2)!} = \frac{n(n-1)}{2}$

Out of $^nC_2$ lines, n are the sides of the polygon and remaining $^nC_2 - n$ are the diagonals

So number of diagonals = $\frac{n(n-1)}{2} - n = \frac{n(n-3)}{2}$

Example : 10

In how many ways can a circket team be selected from a group of 25 players containing 10 batsmen, 8 bowlers, 5 all-rounders and 2 wicketkeepers? Assume that the team of 11 players requires 5 batsmen, 3 all-rounders, 2-bowlers and 1 wicketkeeper.

Solution

Divide the selection of team into four operation.

I : Selection of bastsman can be done (5 from 10) in $^{10}C_5$ ways.

II : Selection of bowlers can be done (2 from 8) in $^8C_2$ ways

III : Selection of all-rounders can be done (3 from 5) in $^5C_3$ ways

IV : Selection of wicketkeeper can be done (1 from 2) in $^2C_1$ ways

$\Rightarrow$ the team can be selected in = $^{10}C_5 \times {}^8C_2 \times {}^5C_3 \times {}^2C_1$ ways = $\frac{10! \times 8 \times 7 \times 10 \times 2}{5!\,5!\,2!} = 141120$

Example : 11

A box contains 5 different red and 6 different white balls. In how many ways can 6 balls be selected so that there are at least two balls of each colour?

Solution

The selection of balls from 5 red and 6 white balls will consist of any of the following possibilities

RED BALLS (out of 5)	WHITE BALLS (out of 6)
2	4
3	3
4	2

- If the selection contains 2 red and 4 white balls, then it can be done in $^5C_2\ ^6C_4$ ways
- If the selection contains 3 red and 3 white balls then it can be done in $^5C_3$ ways
- If the selection contains 4 red and 2 white balls then it can be done in $^5C_4\ ^6C_2$ ways

Any one of the above three cases can occur. Hence the total number of ways to select the balls
$= {}^5C_2\ {}^6C_4 + {}^5C_3\ {}^6C_3 + {}^5C_4\ {}^6C_2$
= 10 (15) + 10 (20) + 5 (15) = 425

Example : 12

How many five-letter words containing 3 vowels and 2 consonants can be formed using the letters of the word E Q U A T I O N so that the two consonants occur together in every word?

Solution

There are 5 vowels and 3 consonants in EQUATION. To form the words we will do there operations.

I. Select vowels (3 from 5) in $^5C_3$ ways

II. Select consonants (2 from 3) in $^3C_2$ ways

III. Arrange the selected letters (3 vowels and 2 consonants always together) in 4! 2! ways. Hence the no. of words = $^5C_3\ ^3C_2$ 4! 2! = 10 × 3 × 24 × 2 = 1440

Example : 13

How many four-letter words can be formed using the letters of the word I N E F F E C T I V E ?

Solution

INEFFECTIVE contains 11 letters : EEE, FF, II, C, T, N, V.
As all letters are not different, we cannot use ^nP_r.
The four-letter words will be among any one of the following categories

(1) 3 alike letters, 1 different letter

(2) 2 alike letters, 2 alike letter

(3) 2 alike letters, 2 different letters

(4) All different letters

(1) 2 alike, 1 different :

3 alike can be selected in one way i.e. EEE
Different letters can be selected from F, I, T, N, V, C in $^6C_1$ ways

⇒ No. of groups = 1 × $^6C_1$ = 6

⇒ No. of words = $6 \times \frac{4!}{3! \times 1!} = 24$

(2) 2 alike 2alike

Two sets of 2 alike can be selected from 3 sets (EE, II, FF) in $^3C_2$ ways

⇒ No. of words = $^3C_2 \times \frac{4!}{2! \times 2!} = 18$

(3) 2 alike 2 different

⇒ No. of groups = $(^3C_1) \times (^6C_2) = 45$

⇒ No, of words = $45 = \frac{4!}{2!} = 540$

(4) All different

⇒ No. of groups = $^7C_4$ (out of E, F, I, T, N, V, C)

Hence total four-letter words = 24 + 18 + 540 + 840 = 1422

Example : 14

A man has 5 friends. In how many ways can he invite one or more of them to a party?

Solution

If he invites one person to the party No. of ways = $^5C_1$

If he invites two persons to the party No. of ways = $^5C_2$

Proceeding on the similar pattern,

Total number of ways to invite = $^5C_1 + {}^5C_2 + {}^5C_3 + {}^5C_4 + {}^5C_5 = 5 + 10 + 10 + 5 + 1 = 31$

Alternate method :

To invite one or more friends to the party, he has to take 5 decisions - one for every friend.

Each decision can be taken in two ways - invited or not invited

Hence (the number of ways to invite one or more)

= (number of ways to make 5 decisions – 1)

= $2 \times 2 \times 2 \times 2 - 1 = 2^5 - 1 = 5!$

Note that we have subtract 1 to exclude the case when all are not invited.

Example : 15

Find the number of ways in which one or more letters can be selected from the letters :

A A A A B B B C D E

Solution

The given letters can be divided into five following categories : (AAAA), (BBB), C, D, E

To select at least one letter, we have to take five decisions-one for every category

Selections from (AAAA) can be made in 5 ways : include no A, include one A, include AA, include AAA, include AAAA

Similarly, selections from (BBB) can be made in 4 ways, and selections from C, D, E can be made in $2 \times 2 \times 2$ ways.

$\Rightarrow$ total number of selections = $5 \times 4 \times (2 \times 2 \times 2) - 1 - 159$

(excluding the case when no letter is selected)

Example : 16

The question paper is an examination contains three sections - A, B, C. There are 6, 4, 3 questions in sections A, V, C respectively. A student has the freedom to answer any number of questions attempting at least one from each section. In how many ways can the paper be attempted by a student?

Solution

There are three possible cases :

(i) Section A contains 6 questions. The student can select at least one from these in $2^6 - 1$ ways.

(ii) Section B contains 4 questions. The student can select at least one from these in $2^4 - 1$ ways.

(iii) Section C can similarly be attempted in $2^3 - 1$ ways

$\Rightarrow$ Hence total number of ways to attempt the paper = $(2^6 - 1)(2^4 - 1)(2^3 - 1) = 63 \times 15 \times 7 = 6615$

Example : 17

Find all number of factors (excluding 1 and the expression itself) of the product of $a^7 b^4 c^3$ d e f where a, b, c, d, e, f are all prime numbers.

Solution

A factor of expression $a^7 b^4 c^3$ d e f is simply the result of selecting one or more letters from 7 a's, 4 b's, 3a's, d, e, f. The collection of letters can be observed as a collection of 17 objects out of which 7 are alike of one kind (a's), 4 are of second kind (b's), 3 are of third kind (c, s) and 3 are different (d, e, f)

The number of selections = $(1 + 7)(1 + 4)(1 + 3) 2^3 = 8 \times 5 \times 4 \times 8 = 1280$.

But we have to exclude two cases :

(i) When no letter is selected

(ii) When all are selected

Hence the number of factors = 1280 – 2 = 1278

Example : 18

In how many ways can 12 books be equally distributed among 3 students ?

Solution

Each student will get 4 books

1. First student can be given 4 books from 12 in $^{12}C_4$ ways

2. Second student can be given 4 books from remaining 8 books in ${}^{8}C_4$ ways

3. Third student can be given 4 books from remaining 4 in ${}^{4}C_4$ ways

$\Rightarrow$ the total number of ways to distribute the books = ${}^{12}C_4 \times {}^{8}C_4 \times {}^{4}C_4$

Example : 19

How many four-letter words can be made using the letters of the word FAILURE, so that

(a) F is included in each word?

(b) F is not included in any word?

Solution

(a) To include F in every word, we will do two operators.

I. Select the remaining three letters from remaining 6 letters i.e. A, I, L, U, R, E in ${}^{7-1}C_{4-1} = {}^{6}C_3$ ways

II. Include F in each group and then arrange each group of four letters in 4! ways

No. of words = ${}^{6}C_3$ 4! = 480

(b) If F not to be included, then we have to select all the four letters from the remaining 6.

No. of words = ${}^{7-1}C_4$ 4! = ${}^{6}C_4$ 4! = 360

Example : 20

(a) In how many ways can 5 persons be arranged around a circular table ?

(b) In how many of these arrangements will two particular persons be next to each other?

Solution

(a) Let the five persons be A_1, A_2, A_3, A_4, A_5 .

Let us imagine A_1 as fixed in its position. The remaining 4 persons can be arranged among themselves in 4! ways.

Hence the number of different arrangements = (5 – 1)! = 4! = 24

(b) Let us assume that A_1 and A_2 are the two particular persons next to each other.

Treating $(A_1 A_2)$ as one person, we have 4 persons in all to arrange in a circle : $(A_1 A_2) A_3 A_4 A_5$. These can be arranged in a circle in (4 – 1)! = 3! = 6 ways.

Now A_1 and A_2 can be arranged among themselves in 2! ways.

Hence total number of arrangement = 6 × 2! = 12

Example : 21

There are 20 persons among whom are two brothers. Find the number of ways in which we can arrange them around a circle so that there is exactly one person between the two brothers.

Solution

Let B_1 and B_2 are the two brother and M be a person sitting between B_1 and B_2 .

Divide this problem into three operations.

I. Select M from 18 persons (excluding B_1 and B_2). This can be done in ${}^{18}C_1$ ways

II. Treat (B_1, M, B_2) as one person. Arrange (B_1, M, B_2) and other 17 persons around a circle. This can be done in (18 – 1) ! = 17 ! ways

III. B_1 and B_2 can be arranged among themselves in 2! ways.

So total number of ways = ${}^{18}C_1 \times 17! \times 2 = 2\ (18!)$

Example : 22

Three tourist want to stay in five different hotels. In how many ways can they do so if :

(a) each hotel cannot accommodate more than one tourist?

(b) each hotel can accommodate any number of tourists?

Solution

(a) There tourists are to be placed in 3 different hotels out of 5. This can be done in two steps :

I. Select three hotels from five in ${}^{5}C_3$ ways.

II. Place the three tourists in 3 selected hotels in 3! ways

$\Rightarrow$ the required number of ways = ${}^{5}C_3$ 3! = 5 × 4 × 3 = 60

(b) To place the tourists we have to do following three operations

I. Place first tourist in any of the hotels in 5 ways ways

II. Place second tourist in any of the hotels in 5 ways

III. Place third tourist in any of the hotels in 5 ways

$\Rightarrow$ the required number of placements = 5 × 5 × 5 = 125

Example : 23

How many seven-letter words can be formed by using the letters of the word S U C C E S S so that :

(a) the two C are together but not two S are together?

(b) no two C and no two S are together

Solution

(a) Considering CC as single object U, CC, E can be arranged in 3! ways.

X U X C C X E X

Now the three S are to be placed in the 4 available places (X) so that CC are not separated but S are separated.

No. of ways to place SSS = (No. of ways to select 3 places) x 1 = $^4C_3$ x 1 = 4

$\Rightarrow$ No. of words = 3! x 4 = 24

(b) Let us first find the words in which no two S are together. To achieve this, we have to do following operations.

(i) Arrange the remaining letters UCCE in $\frac{4!}{2!}$ ways

(ii) Place the three SSS in any arrangement from (i)

X U X C X C X E X

There are five available places for three SSS.

No. of placements = $^5C_3$

Hence total number of words with no two S together = $\frac{4!}{2!}\ ^5C_3 = 120$

No. of words having CC separated and SSS separated = (No. of words having SSS separated) - (No. of words having SSS separated but CC together = 120 – 24 = 96 [using result of part (a)]

Example : 24

A ten party is arranged for 16 people along two sides of a long table with 8 chairs on each side. Four men wish to sit on one particular side and two on the other side. In how many ways can they be seated?

Solution

Let A_1, A_2, A_3, ... , A_{16} be the sixteen persons. Assume that A_1, A_2, A_3, A_4 want to sit on side 1 and A_5, A_6 wan to sit on side 2.

The persons can be made to sit if we complete the following operations.

(i) Select 4 chairs from the side 1 in $^8C_4$ ways and allot these chairs to A_1, A_2, A_3, A_4 in 4! ways

(ii) Select two chairs from side 2 in $^8C_2$ ways and allot these two chairs to A_5 , A_6 in 2! ways

(iii) Arrange the remaining 10 persons in remaining 10 chairs in 10 ! ways

$\Rightarrow$ Hence the total number of ways in which the persons can be arranged

$$= (^8C_4\ 4!)\ (^8C_2\ 2!)\ (10!) = \frac{8!}{4!\ 4!}\ 4! \times \frac{8!\ 2!}{2!\ 6!}\ 10! = \frac{8!\ 8!\ 10!}{4!\ 6!}$$

Example : 25

A mixed doubles tennis game is to be arranged from 5 married couples. In how many ways the game be arranged if no husband and wife pair is included in the same game?

Solution

To arrange the game we have to do the following operating

(i) Select two men from from 5 men in $^5C_2$ ways.

(ii) Select two women from 5 women excluding the wives of the men already selected. This can be done in $^3C_2$ ways.

(iii) Arrange the 4 selected persons in two teams. If the selected men are M_1 and M_2 and the selected women are W_1 and W_2 , this can be done in 2 ways :

$M_1\ W_1$ play against $M_2\ W_2$

$M_2\ W_1$ play against $M_1\ W_2$

Hence the number of ways to arrange the game = $^5C_2\ ^3C_2\ (2) = 10 \times 3 \times 2 = 60$

Example : 26

A man has 7 relatives, 4 of them ladies and 3 gentlemen; his wife has 7 relatives, 3 of them are ladies and 4 gentlemen. In how many ways can they invite a dinner party of 3 ladies and 3 gentlemen so that there are 3 of man's relatives and 3 of wife's relatives?

Solution

The possible ways of selecting 3 ladies and 3 gentleman for the party can be analysed with the help of the following table.

Man's relative		**Wife's relative**		**Number of ways**
Ladies (4)	Gentlemen (3)	Ladies (3)	Gentlemen (4)	
3	0	0	3	${}^4C_3\ {}^3C_0\ {}^3C_0\ {}^4C_3 = 16$
2	1	1	2	${}^4C_2\ {}^3C_1\ {}^3C_1\ {}^4C_2 = 324$
1	2	2	1	${}^4C_1\ {}^3C_2\ {}^3C_2\ {}^4C_1 = 144$
0	3	3	0	${}^4C_0\ {}^3C_3\ {}^3C_3\ {}^4C_0 = 1$

Total number of ways in invite = 16 + 324 + 144 + 1 = 485

Example : 27

In how many ways can 7 plus (+) signs and 5 minus (–) signs be arranged in a row so that no two minus (–) signs are together?

Solution

(i) The plus signs can be arranged in one way (because all are identical)

	+		+		+		+		+		+		+

A blank box shows available spaces for the minus signs.

(ii) The 5 minus (–1) signs are now to be placed in the 8 available spaces so that no two of them are together

(i) Select 5 places for minus signs in ${}^8C_5$ ways.

(ii) Arrange the minus signs in the selected places in 1 way (all signs being identical).

Hence number of possible arrangements = $1 \times {}^8C_5 \times 1 = 56$

Example : 28

There are p points in a plane, no three of which are in the same straight line with the exception of q, which are all in the same straight line. Find the number of

(a) straight lines,

(b) triangles

which can be formed by joining them.

Solution

(a) If no three of the p points were collinear, the number of straight lines = number of groups of two that can be formed from p points = ${}^pC_2$.

Due to the q points being collinear, there is a loss of ${}^qC_2$ lines that could be formed from these points.

But these points are giving exactly one straight line passing through all of them.

Hence the number of straight line = ${}^pC_2 - {}^qC_2 + 1$

(b) If no three points were collinear, the number of triangles = ${}^pC_3$

But there is a loss of ${}^qC_2$ triangles that could be formed from the group of q collinear points.

Hence the number of triangles formed = ${}^pC_3 - {}^qC_3$

Example : 29

(a) How many six-digit numbers can be formed using the digits 0, 1, 2, 3, 4, 5?

(b) How many of these are even?

(c) How many of these are divisible by 4?

(d) How many of these are divisible by 25?

Solution

(a) To make six digit number we have to fill six places. The corresponding choices are as follows:

Places						
No. of choices	5	5	4	3	2	1

$\Rightarrow$ $5 \times 5! = 600$ numbers.

(b) To calculate even numbers, we have to count in two parts :

(i) even number ending in 0

Places						0
No. of choices	5	4	3	2	1	1

$\Rightarrow$ $5! = 120$ numbers can be formed

(ii) even numbers ending in 2, 4

Places						2, 4
No. of choices	4	4	3	2	1	2

There are two choices (2, 4) for the last place and four choices (non-zero digit from remaining) for the first places.

$\Rightarrow$ $4 \times 4! \times 2 = 192$ numbers can be formed. Hence total even numbers that can be formed $= 120 + 192 = 312$

(c) The multiples of 4 can be divided into following groups

(i) ending with (04)

Places					0	4
No. of choices	4	3	2	1	1	1

$\Rightarrow$ $4! = 24$ multiples of 4 ending in (04) are possible

(ii) ending with (24)

Places					2	4
No. of choices	3	3	2	1	1	1

There are 3 choices (1, 3, 5) for the first place. Remaining three places can be filled in 3! ways using any of the remaining three digits

$\Rightarrow$ $3 \times 3! = 18$ numbers are possible

(iii) ending with 0

Places					2, 4	0
No. of choices	4	3	2	1	2	1

Note that there are two choices (2, 4) for the ten's place.

$\Rightarrow$ $4! \times 2 = 48$ numbers are possible

(iv) ending with 2

Places					1, 3, 5	2
No. of choices	3	3	2	1	3	1

Note that there are three choices (1, 3, 5) for the ten's place

$\Rightarrow$ $3 \times 3! \times 3 \times 1 = 54$ numbers are possible

Hence the total number of multiples of 4 = 24 + 48 + 72 = 144

(d) numbers divisible by 25 must end with 25 or 50

(i) ending with 2,5

Places					2	5
No. of choices	3	3	2	1	1	1

$\Rightarrow$ $3 \times 3! = 18$ numbers are possible

(ii) ending with 5, 0

Places					5	0
No. of choices	4	3	2	1	1	1

$\Rightarrow$ $4! = 24$ numbers are possible

Hence total numbers of multiples of 25 = 18 + 24 = 42

Example : 30

Find the sum of all five-digit numbers that can be formed using digits 1, 2, 3, 4, 5 if repetition is not allowed?

Solution

There are 5! = 120 five digit numbers and there are 5 digits. Hence by symmetry or otherwise we can see that each digit will appear in any place (unit's or ten's or ....) $\frac{5!}{5}$ times

$\Rightarrow$ X = sum of digits in any place

$$X = \frac{5!}{5} \times 5 + \frac{5!}{5} \times 4 + \frac{5!}{5} \times 3 + \frac{5!}{5} \times 2 + \frac{5!}{5} \times 1$$

$$X = \frac{5!}{5} \times (5 + 4 + 3 + 2 + 1) = \frac{5!}{5}\ (15)$$

$\Rightarrow$ the sum of all numbers = X + 10 X + 100X + 1000X + 10000X

= X(1 + 10 + 100 + 1000 + 10000)

$$= \frac{5!}{5}\ (15)\ (1 + 10 + 100 + 1000 + 10000)$$

= 24 (15) (11111) = 3999960

Example : 31

Find the number of ways of distributing 5 identical balls three boxes so that no box is empty and each box being large enough to accommodate all the balls.

Solution

Let x_1, x_2 and x_3 be the number of balls places in Box – 1, Box - 2 and Box - 3 respectively.

The number of ways of distributing 5 balls into Boxes 1, 2 and 3 is the number of integral solutions of the equation $x_1 + x_2 + x_3 = 5$ subjected to the following conditions on x_1, x_2, x_3 ....(i)

Conditions on x_1, x_2 and x_3

According to the condition that the boxes should contain atleast one ball, we can find the range of x_1, x_2 and x_3 i.e.

Min. $(x_i) = 1$ and Max $(x_i) = 3$ for $i = 1, 2, 3$

[using : Max $(x_i) = 5 -$ Min $(x_2) -$ Min (x_3)]

or $1 \le x_i \le 3$ for $i = 1, 2, 3$

So, number of ways of distributing balls

= number of integral solutions of (i)

= coeff. of x^5 in the expansion of $(x + x^2 + x^3)^3$

= coeff. of x^2 in $(1 - x^3)(1 - x)^{-3}$

= coeff. of x^2 in $(1 - x)^{-3}$

$= {}^{3+2-1}C_2 = 6$

Alternate solution

The number of ways of dividing n identical objected into r groups so that no group remains empty

$= {}^{n-1}C_{r-1}$ [using result 6.3(a)]

$= {}^{5+1}C_{3-1} = {}^{4}C_{2} = 6$

Example : 32

Find the number of ways of distributing 10 identical balls in 3 boxes so that no box contains more than four balls and less than 2 balls

Solution

Let x_1, x_2 and x_3 be the number of balls placed in Boxes 1, 2 and 3 respectively

Number of ways of distributing 10 balls in 3 boxes = Number of integral solutions of the equation $x_1 + x_2 + x_3 = 10$ ....(i)

Conditions on x_1, x_2 and x_3

As the boxes should contain atmost 4 ball, we can make $Max(x_i) = 4$ and $Min\ (x_i) = 2$ for i = 1, 2, 3

[using : $Min\ (x_1) = 10 - Max\ (x_2) - Max\ (x_3)$]

or $2 \le x_i \le 4$ for i = 1,2 , 3

So the number of ways of distributing balls in boxes = number of integral solutions of equation (i)

= coeff. of x^{10} in the expansion of $(x^2 + x^3 + x^4)^3$

= coeff. of x^{10} in $x^6 (1 - x^3)^3 (1 - x)^{-3}$

= coeff. of x^4 in $(1 - x^3)^3 (1 - x)^{-3}$

= coeff. of x^4 in $(1 - {}^3C_1 x^3 + {}^3C_2 x^6 + ...) (1 - x)^{-3}$

= coeff. of x^4 in $(1 - x)^{-3}$ – coeff. of x in ${}^3C_1 (1 - x)^{-3}$

$= {}^{4+3-1}C_4 - 3 \times {}^{3+1-1}C_1 = {}^6C_4 - 3 \times {}^3C_1 = 15 - 9 = 6$

Note : Instead of taking minimum value $x_i = 2$

(for i = 1, 2, 3), we can also consider it 0 i.e. we can take $0 \le x_i \le 4$

Example : 33

In a box there are 10 balls, 4 are red, 3 black, 2 white and 1 yellow. In how many ways can a child select 4 balls out of these 10 balls? (Assume that the balls of the same colour are identical)

Solution

Let x_1, x_2 x_3 and x_4 be the number of red, black, white, yellow balls selected respectively

Number of ways to select 4 balls = Number of integral solution of the equation $x_1 + x_2 + x_3 + x_4 = 4$

Conditions on x_1, x_2, x_3 and x_4

The total number of red, black, white and yellows balls in the box are 4, 3, 2 and 1 respectively.

So we can take :

$Max\ (x_1) = 4$, $Max\ (x_2) = 3$, $Max\ (x_3) = 2$, $Max\ (x_1) = 1$

There is no condition on minimum number of red, black, white and yellow balls selected, so take :

$Min\ (x_i) = 0$ for 1 = 1, 2, 3, 4

Number of ways to select 4 balls = coeff. of x^4 in

$(1 + x + x^2 + x^3 + x^4) \times (1 + x + x^2 + x^3) \times (1 + x + x^2) \times (1 + x)$

= coeff. of x^4 in $(1 - x^5) (1 - x^4) (1 - x^3) (1 - x^2) (1 - x)^{-4}$

= coeff. of x^4 in $(1 - x)^{-4}$ – coeff. of x^2 in $(1 - x)^{-4}$ – coeff. of x^1 in $(1 - x)^{-4}$ – coeff. off x^0 in $(1 - x)^{-4}$

$$= {}^7C_4 - {}^5C_2 - {}^4C_1 - {}^3C_0 = \frac{7 \times 6 \times 5}{3!} - 10 - 4 - 1 = 35 - 15 = 20$$

Thus, number of ways of selecting 4 balls from the box subjected to the given conditions is 20.

Alternate Solutions :

The 10 balls are RRRR BBB WW Y (where R, B, W, Y represent red, black, white and yellow balls respectively).

The work of selection of the balls from the box can be divided into following categories

Case – 1 All alike

Number of ways of selecting all alike balls = ${}^1C_1 = 1$

Case – 2 3 alike and 1 different

Number of ways of selecting 3 alike and 1 different balls = ${}^2C_1 \times {}^3C_1 = 6$

Case – 3 2 alike and 2 alike

Number fo ways of selecting 2 alike and 2 alike balls = ${}^3C_2 = 3$

Case – 4 2 alike and 2 different

Number of ways of selecting 2 alike and 2 different balls = ${}^3C_1 \times {}^3C_1 = 9$

Case – 5 All different

Number of ways of selecting all different balls = ${}^4C_4 = 1$

Total number of ways to select 4 balls = 1 + 6 + 3 + 9 + 1 = 20

Example : 34

A person writes letters to 4 friends and addresses the corresponding envelopes. In how many ways can the letters be placed in the envelopes so that :

(i) atleast two of them are in the wrong envelopes

(ii) all the letters are in the wrong envelopes

Solution

(i) Number of ways to place 4 letters in 4 envelopes without any condition = 4!

Number of ways to place all letters correctly into the corresponding envelopes = 1

Number of ways to place one letter is the wrong envelop and other 3 letters in the write envelope = 0

(Because it is not possible that only one letter goes in the wrong envelop)

Number of ways to place atleast two letters in the wrong envelopes

= Total number of way to place letters

– Number of ways to place all letters correctly

– Number of ways to place on letter correctly = 4! – 1 – 0 = 23

(ii) Number of ways to put 2 letters in 2 addressed envelopes so that all are in the wrong envelopes = 1.

Number fo ways to put 3 letters in 3 addressed envelopes so that all are in the wrong envelopes = Number of ways without restriction – Number of ways in which all letters are in the correct envelopes – Number of ways in which 1 letter is in the correct envelope = $3! - 1 - 1 \times {}^3C_1 = 2$.

(${}^3C_1$ means that select one envelop to put the letter correctly)

Number of ways to put 4 letters in 4 addressed envelopes so that all are in the wrong envelopes

= Number of ways without restriction

– Number of ways in which all letters are in the correct envelopes

– Number of ways in which 1 letter is in the correct envelopes

(i.e. 3 are in the wrong envelopes)

– Number of ways in which 2 letters are in the correct envelopes

(i.e. 2 are in the wrong envelopes)

$= 4! - 1 - 4C_1 \times 2 - 4C_2 \times 1 = 24 - 1 - 8 - 6 = 9$

Alternate Solution :

Use result 6.4 (e)

The required number of ways to place all 4 letters in the wrong envelopes

$$= 4!\left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!}\right) = 24\left(1 - 1 + \frac{1}{2} - \frac{1}{6} + \frac{1}{24}\right) = 9$$

Example : 35

Find the number of ways of distributing 5 different balls in there boxes of different sizes so that no box is empty and each box being large enough to accommodate all the five balls.

Solution

Method – 1

The five balls can be distributed in 3 non-identical boxes in the following 2 ways :

Boxes	**Box 1**	**Box 2**	**Box 3**
Number of balls	**3**	**2**	**1**
No. of choices	**2**	**2**	**1**

Case – 1 : 3 in one Box, 1 in another and 1 in third Box (3, 1, 1) ....(i)

Number of ways to divide balls corresponding to (1) = $\frac{5!}{3!\,1!\,1!}\ \frac{1}{2!}$ = 10

But corresponding to each division there are 3! ways of distributing the balls into 3 boxes.
So number of ways of distributing balls corresponding to
(i) = (No. of ways to divide balls) × 3! = 10 × 3! = 60
Case – 2 : 2 in one Box, 2 in another and 1 in third Box (2, 2, 1)

Number of ways to divide balls corresponding to (2) = $\frac{5!}{2!\,2!\,1!}\ \frac{1}{2!}$ = 15

But corresponding to each division there are 3! ways of distributing of balls into 3 boxes.
So number of ways of distributing balls corresponding to
(2) = (No. of ways to divide balls) × 3! = 15 × 3! = 90
Hence required number of ways = 60 + 90 = 150
Method – 2
Let us name to Boxes as A, b and C. Then there are following possibilities of placing the balls :

Box A	**Box B**	**Box C**	**Number of ways**
1	**2**	**2**	$^5C_1 \times {}^4C_2 \times {}^2C_2 = 30$
1	**1**	**3**	$^5C_1 \times {}^4C_1 \times {}^3C_3 = 20$
1	**3**	**1**	$^5C_1 \times {}^4C_3 \times {}^1C_1 = 20$
2	**1**	**2**	$^5C_2 \times {}^3C_1 \times {}^2C_2 = 30$
2	**2**	**1**	$^5C_2 \times {}^3C_2 \times {}^1C_1 = 30$
2	**1**	**1**	$^5C_3 \times {}^2C_1 \times {}^1C_1 = 20$

Therefore required number of ways of placing the balls = 30 + 20 + 20 + 30 + 30 + 20 = 150
Method – 3
Using Result 6.2 (a)
Number of ways of distributing 5 balls in 3 Boxes so that no Box is empty = $3^5 - 3 \times 2^5 + 3 \times 1^5$ = 150.

Example : 36

If n distinct objects are arranged in a circle, show that the number of ways of selecting three of these things o that no two of them are next to each other is $\frac{n}{6}$ (n – 4) (n – 5).

Solution

Let a_1, a_2, a_3 ...., be the n distinct objects
Number of ways to select three objects so that no two of them are consecutive = Total number of ways to select three objects – Number of ways to select three consecutive objects – Number of ways to select three objects in which two are consecutive and one is separated ....(i)
Total number of ways to select 3 objects from n distinct objects = $^nC_3$ ....(ii)
Select three consecutive objects
The three consecutive objects can be selected in the following manner
Select from :
$a_1\, a_2\, a_3$, $a_2\, a_3\, a_4$, $a_3\, a_4\, a_5$, ..., $a_{n-1}\, a_n\, a_1$, $a_n\, a_1\, a_2$
So number of ways in which 3 consecutive objects can be selected from n objects arranged in a circle in n. ....(ii)
Select two consecutive (together) and 1 separated
The three objects so that 2 are consecutive and 1 is separated can be selected in the following manner :
Take $a_1\, a_2$ ad select third object from a_4, a_5 , ..., a_{n-1}
i.e. take $a_1\, a_2$ and select third object in (n – 4) ways or in general we can say that select one pair from n available pairs i.e. a_1, a_2, a_2, a_3 , ..., $a_n\, a_1$ and third object in (n – 4) ways

So number of ways to select 3 objects so that 2 are consecutive and 1 is separated = n (n – 4)
.......(iv)

Using (i), (ii), (iii) and (iv), we get

Number of ways to select 3 objects so that all are separated

$$= {}^{n}C_{3} - n - n\,(n-4) = \frac{n(n-1)(n-2)}{6} - n - n\,(n-4) = n\left[\frac{n^2 - 3n + 2 - 6(n-3)}{6}\right]$$

$$= \frac{n}{6}\,(n^2 - 9n + 20) = \frac{n}{6}\,(n-4)\,(n-5)$$

Example : 37

Find the number of integral solutions of the equation 2x + y + z = 20 where x, y, z ≥ 0

Solution

2x = y + z = 20 ....(i)

Condition on x

x is maximum when y and z are minimum

$\Rightarrow$ 2 Max (x) = 20 – Min (y) – Min (x)

$\Rightarrow$ Max (x) = $\frac{20-0-0}{2}$ = 10

Let x = k where 0 ≤ k ≤ 10

Put x = k in (i) to get, y + z = 20 – 2k ....(ii)

Number of non-negative integral solutions of (ii) = 20 – 2k + 1 = 2k – 2k

As k is varies from 0 to 10, the total number of non-negative integral solutions of (1)

$$= \sum_{k=0}^{10} (21 - 2k) = \sum_{k=0}^{10} 21 = 2\sum_{k=0}^{10} k = 231 - 110 = 121$$

$$\left(\text{using} \sum n = \frac{n(n+1)}{2}\right)$$

Hence, total number of non-negative integral solutions of (i) is 121

Example : 38

These are 12 seats in the first row of a theater of which 4 are to be occupied. Find the number of ways of arranging 4 persons so that :

(i) no two persons sit side by side

(ii) there should be atleast 2 empty seats between any two persons

(iii) each person has exactly one neighbour

Solution

(i) We have to select 4 sets for 4 persons so that no two persons are together. It means that there should be atleast one empty seat vacent between any two persons.

To place 4 persons we have to select 4 seats between the remaining 8 empty seats so tat all persons should be separated.

Between 8 empty seats 9 seats are available for 4 person to sit.

Select 4 seats in ${}^{9}C_{4}$ ways

But we can arrange 4 persons on these 4 seats in 4! ways. So total number of ways to give seats to 4 persons so that no two of them are together = ${}^{9}C_{4}$ = 4! = ${}^{9}P_{4}$

(ii) Let x_0 denotes the empty seats to the left of the first person, x_i (i = 1, 2, 3) be the number of empty seats between i th and (i + 1) st person and x_4 be the number of empty seats to the right of 4th person.

Total number of seats are 12. So we can make this equation : $x_0 + x_1 + x_2 + x_3 + x_4 = 8$ ......(i)

Number of ways to give seats to 4 persons so that there should be two empty seats between any two persons is same as the number of integral solutions of the equation (i) subjected to the following conditions.

Conditions on x_1, x_2, x_3, x_4

According to the given condition, these should be two empty seats between any two persons i.e.

Min $(x_i) = 2$ for i = 1, 2, 3

Min $(x_0) = 0$ and Min $(x_0) = 0$

Max $(x_0) = 8 -$ Min $(x_1 + x_2 + x_3 + x_4) = 8 - (2 + 2 + 2 - 0) = 2$

Max $(x_4) = 8 -$ Min $(x_0 + x_1 + x_2 + x_3) = 8 - (2 + 2 + 2 - 0) = 2$

Similarly,

Max (xi) = 4 for i = 1, 2, 3

No. of integral solutions of the equation (i) subjected to the above conditions = coeff. of x^8 in the expansion of $(1 + x + x^2)^2\ (x^2 + x^3 + x^4)^3$ = coeff. of x^8 in $x^6\ (1 + x + x^2)^5$ = coeff of x^2 in $(1 - x^3)^5\ (1 - x)^{-5}$ = coeff of x^2 in $(1 - x)^{-5} = {}^{5+2-1}C_2 = {}^6C_2 = 15$

Number of ways to select 4 seats so that there should be atleast two empty seats between any two persons = 15.

But 4 persons can be arranged in 4 seats in 4! ways.

So total number of ways to arrange 4 persons in 12 seats according to the given condition $= 15 \times 4! = 360$

(iii) As every person should have exactly one neighbour, divide 4 persons into groups consisting two persons in each group.

Let G_1 and G_2 be the two groups in which 4 persons are divided.

According to the given condition G_1 and G_2 should be separated from each other.

Number of ways to select seats so that G_1 and G_2 are separated ${}^{8+1}C_2 = {}^9C_2$

But 4 persons can be arranged in 4 seats in 4! ways. So total number of ways to arrange 4 persons so that every person has exactly one neighbout $= {}^9C_2 \times 4! = 864$

Example : 39

The number of non-negative integral solutions of $x_1 + x_2 + x_3 + x_4 \le n$ where n is a positive integer

Solution

It is given that : $x_1 + x_2 + x_3 + x_4 \le 4$ ....(i)

Let $x_5 \ge 0$

Add x_5 on LHS of (i) to get $x_1 + x_2 + x_3 + x_4 + x_5$ ....(ii)

Number of non-negative integral solutions of the inequation (i) = Number of non-negative integral solutions of the equation (ii)

= coeff. of x^n in $(1 + x + x^2 + x^3 + x^4 + ... + x^n)^5$

= coeff. of x^n in $(1 - x^{n+1})^5\ (1 - x)^{-5}$

= coeff. of x^n in $(1 - x)^{-5} = {}^{n+5+1}C_n = {}^{n+4}C_n = {}^{n+4}C_n$.

Example : 40

If all the letters of the word RANDOM are written in all possible orders and these words are written out as in a dictionary, then find the rank of the word RANDOM in the dictionary.

Solution

In a dictionary the words at each stage are arranged in alphabetical order. In the given problem we must therefore consider the words beginning with A, D, M, N, O, R in order. A will occur in the first place as often as there are ways of arranging the remaining 5 letters all at a line i.e. A will occur 5! times. D, M, N, O will occur in the first place the same number of times.

Number of words starting with A = 5! = 120

Number of words starting with D = 5! = 120

Number of words starting with M = 5! = 120

Number of words starting with N = 5! = 120

Number of words starting with O = 5! = 120

After this words beginning with RA must follow

Number of words beginning with RAD or RAM = 3!

Now the words beginning with RAN must follow
First one is RANDMO and the next one is RANDOM.
$\therefore$ Rank of RANDOM = 5(5!) + 2 (3!) + 2 = 614

Example : 41

What is the largest integer n such that 33! divisible by 2^n?

Solution

We know that : 33! = 1 × 2 × 3 × 4 .... × 32 × 33

$\Rightarrow$ 33! = (2 × 4 × 6 .... × 32) (1 × 3 × 5........ × 33)

$\Rightarrow$ 33! = 2^{16} (1 × 2 × 3 × 4 ...... × 15 × 16) (1 × 3 × 5 ...... × 33)

$\Rightarrow$ 33! = 2^{16} (2 × 4 .... × 16) (1 × 3 ..... × 15) (1 × 3 ....×33)

$\Rightarrow$ 33! = 2^{16} (1 × 2 × .... × 8) (1 × 3 × .... × 15) (1 × 3 ...... × 33)

$\Rightarrow$ 33! = 2^{24} (2 × 4 × 6 × 8) (1 × 3 × 5 × 7) (1 × 3 × ..... × 15) *1 × 3 × ....× 33)

$\Rightarrow$ 33! = 2^{24} . 2^4 (1 × 2 × 3 × 4) (1 × 3 × 5 × 7) (1 × 3 .... × 15) (1 × 3 .... × 33)

$\Rightarrow$ 33! = 2^{28} (2 × 4) (1 × 3) (1 × 3 × 5 × 7) (1 × 3 × .... × 15) (1 × 3 × ...... × 33)

$\Rightarrow$ 33! = 2^{31} (1 × 3) (1 × 3 × 5 × 7) (1 × 3 × .... × 15) (1 × 3 × .... × 33)

Thus the maximum value of n for which 33! is divisible by 2^n is 31

Example : 42

Find the sum of all four digit numbers formed by using the digits 0, 1, 2, 3, 4, no digits being repeated in any number.

Solution

Required sum of number = (sum of four digit number using 0, 1, 2, 3, 4, allowing 0 in first place) – (Sum of three digit numbers using 1, 2, 3, 4)

$= \frac{5!}{5}$ (0 + 1 + 2 + 3 + 4) (1 + 10 + 10^2 + 10^3) – $\frac{4!}{4}$

(i.e. 3 are in the wrong envelopes) (1 + 2 + 3 + 4) (1 + 10 + 10^2)

= 24 × 10 × 1111 – 6 × 10 × 111 = 259980

Example : 43

In how many ways three girls and nine boys can be seated in two vans, each having numbered seats, 3 in the front and 4 at the back? How many seating arrangements are possible if 3 girls should sit together in a back row on adjacent seats?

Solution

(i) Out of 14 seats (7 in each Van), we have to select 12 seats for 3 girls and 8 boys
12 seats from 14 available seats can be selected in ${}^{14}C_{12}$ ways
Now on these 12 seats we can arrange 3 girls and 9 boys in 12! ways
So total number fo ways = ${}^{14}C_{12}$ × 12! = 91

(ii) One Van out of two available can be selected in ${}^{2}C_{1}$ways
Out of two possible arrangements (see figure) of adjacent seats, select one in ${}^{2}C_{1}$ ways
Out of remaining 11 seats, select 9 for 9 boys in ${}^{11}C_{9}$ ways
Arrange 3 girls on 3 seats in 3! ways and 9 boys on 9 seats in 9! ways
So possible arrangement of sitting (for 3 girls and 9 boys in 2 Vans)
= ${}^{2}C_{1}$ × ${}^{2}C_{1}$ × ${}^{11}C_{9}$ × 3! × 9! = 12! ways

Example : 44

Show that the number of ways of selecting n things out of 2n things of which na re of one kind and alike and n are of a second kind and alike and the rest are unlike is (n + 2) 2^{n-1} .

Solution

Let group G_1 contains first n similar things, group G_2 contains next n similar things let D_1, D_2, D_3, ..., D_n be the n unlike things.

Let x_1 be the number of things selected from group G_1, x_2 be the number of things selected fro group G_2 and p_1, p_2, p_3 ,........., p_n be the number of things selected from D_1, D_2, D_3 , ..., D_n respectively.

As we have to select n things in all, we can make $x_1 + x_2 + p_1 + p_2 + ... + p_n = n$ ....(i)

Number of ways to select n things = Number of integral solutions of the equation (i) subjected to following conditions

Conditions on the variables

There is no condition on the number of items selected from group G_1 and G_2 . So we can take :
Min (x_1) = Min (x_2) = 0 and Max (x_1) = Max (x_2) = n
For items D_1 to D_n , we can make selection in two ways. That is either we take the item or we reject the item. So we can make :

Min $(P_i) = 0$ for $i = 1, 2, 3 \ldots\ldots, n$ and
Max $(p_i) = 1$ for $i = 1, 2, 3 \ldots\ldots\ldots, n$

Find solutions

Number of integral solutions of (1)
$= \text{coeff. of } x^n \text{ in } (x^0 + x^1 + \ldots\ldots\ldots + x^n)^2 (1 + x) (1 + x) \ldots\ldots\ldots n \text{ times}$
$= \text{coeff. of } x^n \text{ in } (1 - x^{n+1})^2 (1 - x)^{-2} (1 + x)^n$
$= \text{coeff. of } x^n \text{ in } (1 - x)^{-2} [2 - (1 - x)]^n$
$= \text{coeff. of } x^n \text{ in } (1 - x)^2 + \ldots\ldots\ldots + {}^nC_n 2^0 (1 - x)^n = \text{coeff. of } x^n \text{ in } [{}^nC_0 2^n (1 - x)^{-2} - {}^nC_1 2^{n-1} (1 - x)^{-1}]$
(because other terms can not product x^n)
$= 2^n \times {}^{n+2-1}C_n - n\, 2^{n-1} \times {}^{2+1-1}C_n = (n + 1)\, 2^n - n\, 2^{n-1} = (n + 2)\, 2^{n-1}$