

Example : 1

If a, b, c are the x th, y th and z th terms of an A.P., show that :

- (a) $a(y - z) + b(z - x) + c(x - y) = 0$
- (b) $x(b - c) + y(c - a) + z(a - b) = 0$

Solution

Let A be the first term and D be the common difference,

$$\Rightarrow T_x = A + (x - 1) D = a \quad \dots\dots\dots(i)$$

$$T_y = A + (y - 1) D = b \quad \dots\dots\dots(ii)$$

$$T_z = A + (z - 1) D = c \quad \dots\dots\dots(iii)$$

operating (2 - 3), (3 - 1) and (1 - 2) we get :

$$b - c = (y - z) D \quad c - a = (z - x) D \quad a - b = (x - y) D$$

$$y - z = \frac{b - c}{D} \quad z - x = \frac{c - a}{D} \quad x - y = \frac{a - b}{D}$$

Now substituting the values of (y - z), (z - x) and (x - y) in L.H.S. of the expression (a) to be proved.

$$\Rightarrow LHS = \frac{a(b - c)}{D} + \frac{b(c - a)}{D} + \frac{c(a - b)}{D}$$

$$\Rightarrow LHS = \frac{ab - ac + bc - ab + ca - cb}{D} = 0 = RHS$$

Now substituting the values of (b - c), (c - a) and (a - b) in LHS of the expression (b) to be proved

$$\Rightarrow LHS = x(y - z) D + y(z - x) D + z(x - y) D$$

$$= \{xy - xz + yz - xz + zx - zy\} D = 0 = RHS$$

Example : 2

The sum of n terms of two series in A.P. are in the ratio 5n + 4 : 9n + 6. Find the ratio of their 13th terms.

Solution

Let a₁, a₂ be the first terms of two A.P.s and d₁, d₂ are their respective common differences.

$$\Rightarrow \frac{\frac{n}{2}[2a_1 + (n - 1) d_1]}{\frac{n}{2}[2a_2 + (n - 1) d_2]} = \frac{5n + 4}{9n + 6}$$

$$\Rightarrow \frac{a_1 + \frac{(n - 1)}{2} d_1}{a_2 + \frac{(n - 1)}{2} d_2} = \frac{5n + 4}{9n + 6} \quad \dots\dots\dots(i)$$

Now the ratio of 13th terms = $\frac{a_1 + 12d_1}{a_2 + 12d_2}$

$$\Rightarrow \text{put } \frac{(n - 1)}{2} = 12 \quad \text{i.e.} \quad n = 25 \text{ in equation (i)}$$

$$\Rightarrow \frac{a_1 + 12d_1}{a_2 + 12d_2} = \frac{5(25) + 4}{9(25) + 6} = \frac{129}{231}$$

Example : 3

If the sum of n terms of a series is S_n = n(5n - 3), find the nth term and pth term.

Solution

$$S_n = T_1 + T_2 + T_3 + T_4 + \dots\dots\dots + T_{n-1} + T_n$$

$$S_n = \{\text{sum of } (n - 1) \text{ terms}\} + T_n$$

$$\Rightarrow T_n = S_n - S_{n-1}$$

Now in the given problem :

$$S_n = n(5n - 3) \text{ and } S_{n-1} = (n - 1)[5(n - 1) - 3]$$

$$\Rightarrow T_n = S_n - S_{n-1} = 10n - 8$$

$$\Rightarrow T_p = 10p - 8$$

Example : 4

If a, b, c, d are in G.P., show that

- (a) $(a - d)^2 = (b - c)^2 + (c - a)^2 + (d - b)^2$
 (b) $a^2 - b^2, b^2 - c^2$ and $c^2 - d^2$ are also in G.P.

Solution

(a) a, b, c, d are in G.P.

$$\Rightarrow b^2 = ac, c^2 = bd, bc = ad \quad \dots\dots\dots(i)$$

Now expanding the RHS, we get

$$\begin{aligned} \text{RHS} &= (b^2 + c^2 - 2bc) + (c^2 + d^2 - 2ac) + (d^2 + b^2 - 2bd) \\ &= 2(b^2 - ac) + 2(c^2 - bd) + a^2 + d^2 - 2bc \\ &= 2(0) + 2(0) + a^2 + d^2 - 2ad \quad (\text{from i}) \\ &= (a - d)^2 = \text{LHS} \end{aligned}$$

(b) Now we have to prove that $a^2 - b^2, b^2 - c^2$ and $c^2 - d^2$ are in G.P. i.e.

$$(b^2 - c^2)^2 = (a^2 - b^2)(c^2 - d^2)$$

Consider the RHS

$$\begin{aligned} (a^2 - b^2)(c^2 - d^2) &= a^2c^2 - b^2c^2 - a^2d^2 + b^2d^2 \\ &= b^4 - b^2c^2 - b^2c^2 + c^4 \\ &= (b^2 - c^2) = \text{LHS} \end{aligned}$$

Example : 5

If a, b, c are in A.P. and x, y, z are in G.P., prove that $x^{b-c} y^{c-a} z^{a-b} = 1$.

Solution

$$a, b, c \text{ are in A.P.} \quad \Rightarrow \quad 2b = a + c \quad \text{or} \quad a - b = b - c$$

$$x, y, z \text{ are in G.P.} \quad \Rightarrow \quad y^2 = xz$$

proceeding from LHS

$$\begin{aligned} &= x^{b-c} z^{b-c} y^{c-a} && \{\text{as } b - c = a - b\} \\ &= (xz)^{b-c} y^{c-a} = y^{2(b-c)} y^{c-a} && \{\text{as } xz = y^2\} \\ &= y^{2(b-c) + (c-a)} \\ &= y^{2b-a-c} = y^0 = 1 = \text{RHS} && \{\text{as } 2b = a + c\} \end{aligned}$$

Example : 6

The sum of three numbers in H.P. is 26 and the sum of their reciprocals is 3/8. Find the numbers.

Solution

Three numbers in H.P. are taken as :

$$\frac{1}{a-d}, \frac{1}{a}, \frac{1}{a+d} \quad \Rightarrow \quad \frac{1}{a-d} + \frac{1}{a} + \frac{1}{a+d} = 26 \quad \dots\dots\dots (i)$$

$$\text{also } (a-d) + a + (a+d) = 3/8 \quad \dots\dots\dots (ii)$$

$$\text{from (i) and (ii) } a = \frac{1}{8} \text{ and } d = \pm \frac{1}{24}$$

\Rightarrow the number are 12, 8, 6 or 6, 8, 12

Example : 7

If pth term of an A.P. is 1/q and the qth term is 1/p. Find the sum of p q terms.

Solution

Let A and D be the first term and the common difference of the A.P.

$$\Rightarrow \frac{1}{q} = A + (p-1)D, \frac{1}{p} = A + (q-1)D$$

solving these two equations to get A and D in terms of p and q

$$\Rightarrow A = \frac{1}{pq} \text{ and } D = \frac{1}{pq}$$

$$\text{sum of } pq \text{ terms} = \frac{pq}{2} [2A + (pq-1)D] = \frac{pq}{2} \left(\frac{2}{pq} + \frac{pq-1}{pq} \right)$$

$$\Rightarrow \text{sum} = \frac{pq+1}{2}$$

Example : 8

If the continued product of three numbers in G.P. is 216 and the sum of the products taken in pairs is 156, find the numbers.

Solution

Let $\frac{b}{r}$, b , br be the three numbers.

$$\Rightarrow \frac{b}{r}, b, br = 216 \quad \Rightarrow \quad b = 6$$

$$\text{also } \frac{b}{r} (b) + b (br) + \frac{b}{r} (br) = 156$$

$$\Rightarrow b^2 \left(\frac{1}{r} + r + 1 \right) = 156 \quad \Rightarrow \quad 6^2 (r^2 + r + 1) = 156r$$

$$\Rightarrow 3r^2 + 3r + 3 = 13r$$

$$\Rightarrow r = 3, \frac{1}{3}$$

Hence the numbers are 2, 6, 18 or 18,, 6, 2

Example : 9

In a HP, the pth term is qr and qth term is rp . Show that the rth term is pq .

Solution

Let A , D be the first term and the common difference of the A.P. formed by the reciprocals of given H.P.

pth term of A.P. is $\frac{1}{qr}$ and qth term of A.P. is $\frac{1}{rp}$

$$\Rightarrow \frac{1}{qr} = A + (p-1)D \text{ and } \frac{1}{rp} = A + (q-1)D$$

We will solve these two equation to get A and D

$$\text{subtracting, we get } \frac{p-q}{pqr} = (p-q)D \quad \Rightarrow \quad D = \frac{1}{pqr}$$

$$\text{Hence } \frac{1}{qr} = A + \frac{p-1}{pqr} \Rightarrow A = \frac{1}{pqr}$$

Now the rth term of A.P. = $T_r = A + (r-1)D$

$$\Rightarrow T_r = \frac{1}{pqr} + \frac{r-1}{pqr} = \frac{1}{pq}$$

Hence rth term of the given H.P. is pq .

Example : 10

The sum of first p , q , r terms of an A.P. are a , b , c respectively. Show that :

$$\frac{a}{p} (q-r) + \frac{b}{q} (r-p) + \frac{c}{r} (p-q) = 0.$$

Solution

Let A be the first term and D be common difference of the A.P.

$$\Rightarrow a = \frac{p}{2} [2A + (p-1)D]$$

We can write $\frac{a}{p}(q-r) + \frac{b}{q}(r-p) + \frac{c}{r}(p-q) = \sum \frac{a}{p}(p-r)$

$$\begin{aligned} \text{LHS} &= \sum \frac{a}{p}(q-r) \\ &= \sum \frac{1}{2}(q-r) [2A + (p-q) d] \\ &= \frac{1}{2} \sum 2A(q-r) + \frac{1}{2} \sum (q-r) D (p-1) \\ &= A \sum (q-r) + \frac{D}{2} \sum [p (q-r)] - \frac{D}{2} \sum (q-r) = 0 + 0 - 0 = 0 = \text{RHS} \end{aligned}$$

Example : 11

If a, b, c are in A.P., then show that $a^2(b+c), b^2(c+a), c^2(a+b)$ are in A.P., if $bc+ca+ab \neq 0$.

Solution

We have to prove that

$$a^2(b+c), b^2(c+a), c^2(a+b) \text{ are in A.P.}$$

i.e. $a(ab+ca), b(bc+ab), c(ca+bc)$ are in A.P.

$$a(ab+bc+ca) - abc, b(bc+ab+ac) - abc, c(ca+bc+ab) - abc \text{ are in A.P.}$$

$$a(ab+bc+ca), b(bc+ab+ca), c(ca+bc+ab) \text{ are in A.P.}$$

$\Rightarrow a, b, c$ are in A.P., which is given.

Hence $a^2(b+c), b^2(c+a), c^2(a+b)$ are in A.P.

Alternate method :

$$\text{As } a, b, c \text{ are in A.P., we get : } a-b = b-c \dots\dots\dots(i)$$

$$\text{Consider } a^2(b+c) - b^2(c+a) = (a^2b - b^2a) + (a^2c - b^2c) = (a-b)(ab+ac+bc) \dots\dots\dots(ii)$$

$$\text{Also } b^2(c+a) - c^2(a+b) = (b^2c - c^2b) + (b^2a - c^2a) = (b-c)(bc+ba+ca) \dots\dots\dots(iii)$$

From (i), (ii), (iii), we get,

$$a^2(b+c) - b^2(c+a) = b^2(c+a) - c^2(a+b)$$

$\Rightarrow a^2(b+c), b^2(c+a), c^2(a+b)$ are in A.P.

Example : 12

If $(b-c)^2, (c-a), (a-b)^2$ are in A.P., then prove that : $\frac{1}{b-c}, \frac{1}{c-a}, \frac{1}{a-b}$ are also in A.P.

Solution

$(b-c)^2, (c-a)^2, (a-b)^2$ are in A.P.

$$\Rightarrow (b-c)^2 - (c-a)^2 = (c-a)^2 - (a-b)^2$$

$$\Rightarrow (b-c)(b+a-2c) = (c-b)(b+c-2a)$$

$$\Rightarrow (b-a)[(b-c) + (a-c)] = (c-a)[(b-a) + (c-a)]$$

Divide by $(a-b)(b-c)(c-a)$

$$\Rightarrow \frac{1}{b-c} - \frac{1}{c-a} = -\frac{1}{a-b} + \frac{1}{c-a}$$

$$\Rightarrow \frac{1}{b-c} - \frac{1}{c-a} = \frac{1}{c-a} - \frac{1}{a-b}$$

$$\Rightarrow \frac{1}{b-c}, \frac{1}{c-a}, \frac{1}{a-b} \text{ are in A.P.}$$

Example : 13

If a, b, c are in G.P., prove that : $\frac{a^2+ab+b^2}{bc+ca+ab} = \frac{b+a}{c+b}$

Solution

As a, b, c are in G.P., let us consider $b = ar$, and $c = ar^2$

$$\text{LHS} = \frac{a^2 + ab + b^2}{b + ca + ab} = \frac{a^2 + a^2r + a^2r^2}{a^2r^3 + a^2r^2 + a^2r} = \frac{a^2(1+r+r^2)}{a^2r(r^2+r+1)} = \frac{1}{r}$$

$$\text{RHS} = \frac{b+a}{c+b} = \frac{ar+a}{ar^2+ar} = \frac{a(r+1)}{ar(r+1)} = \frac{1}{r}$$

Hence LHS = RHS

Example : 14

If a, b, c are respectively pth, qth, rth terms of H.P., prove that $bc(q - c) + ca(r - p) + ab(p - q) = 0$.

Solution

Let A and D be the first term and common difference of the A.P. formed by the reciprocals of the given H.P.

$$\Rightarrow \frac{1}{a} = A + (p - 1) D \quad \dots\dots\dots(i)$$

$$\frac{1}{b} = A + (q - 1) D \quad \dots\dots\dots(ii)$$

$$\frac{1}{c} = A + (r - 1) D \quad \dots\dots\dots(iii)$$

Subtracting II and III we get $\frac{c - b}{bc} = (q - r) D$

$$\Rightarrow bc(q - r) = -\frac{(b - c)}{D}$$

$$\begin{aligned} \text{LHS} &= \sum bc(q - r) \\ &= -\sum \frac{bc}{D} = -\frac{1}{D} \sum (b - c) \\ &= -\frac{1}{D} [b - c + c - a + a - b] = 0 \text{ RHS} \end{aligned}$$

Example : 15

If $a^x = b^y = c^z$ and x, y, z are in G.P., prove that $\log_b a = \log_c b$.

Solution

Consider $a^x = b^y = c^z$. Taking log
 $x \log a = y \log b = z \log c$

$$\Rightarrow \frac{x}{y} = \frac{\log b}{\log a} \text{ and } \frac{y}{x} = \frac{\log c}{\log b}$$

$$\text{as } x, y, z \text{ are in GP} \quad \Rightarrow \quad \frac{x}{y} = \frac{y}{z}$$

$$\Rightarrow \frac{\log b}{\log a} = \frac{\log c}{\log b}$$

$$\Rightarrow \frac{\log a}{\log b} = \frac{\log b}{\log c}$$

$$\Rightarrow \log_b a = \log_c b$$

Example : 16

If $\sqrt[x]{a} = \sqrt[y]{b} = \sqrt[z]{c}$ and if a, b, c are in G.P., then prove that x, y, z are in A.P.

Solution

$$\text{Let } a^{\frac{1}{x}} = b^{\frac{1}{y}} = c^{\frac{1}{z}}$$

$$\Rightarrow \frac{\log a}{x} = \frac{\log b}{y} = \frac{\log c}{z} = k$$

$$\Rightarrow \log a = kx, \quad \log b = ky, \quad \log c = kz$$

$$\text{As } b^2 = ac \Rightarrow 2 \log b = \log a + \log c$$

$$\text{We have } 2ky = kx + kz$$

$$\Rightarrow 2y = x + z$$

$$\Rightarrow x, y, z \text{ are in A.P.}$$

Example : 17

If one GM G and two AM's p and q be inserted between two quantities, show that $G^2 = (2p - q)(2q - p)$.

Solution

Let a, b be two quantities

$$\Rightarrow G^2 = ab \text{ and } a, p, q, b \text{ are in A.P.}$$

$$\Rightarrow p = a + \frac{(b-a)}{3} = \frac{b+2a}{3}, \quad q = a + 2 \frac{b-a}{3} = \frac{2b+a}{3}$$

$$\text{RHS} = (2p - q)(2q - p)$$

$$= \left(\frac{2}{3}(b+2a) - \frac{2b+a}{3} \right) \left(\frac{2(2b+a)}{3} - \frac{b+2a}{3} \right)$$

$$= \frac{1}{9} (2b + 4a - 2b - a) (4b + 2a - b - 2a)$$

$$= \frac{1}{9} (3a) (4b) = ab = G^2 = \text{RHS}$$

Example : 18

If S_n is the sum of first n terms of a G.P. (a_n) and S'_n that of another G.P. ($1/a_n$) then show that :

$$S_n = S'_n = a_1 a_n.$$

Solution

$$S_n = a_1 + a_2 + a_3 + \dots + a_n$$

$$S'_n = \frac{1}{a_1} + \frac{1}{a_2} + \dots + \frac{1}{a_n}$$

For the first G.P. (a_n), $a_n = a_1 r^{n-1}$

$$S_n = \frac{a_1(1-r^n)}{1-r}, \text{ where } r \text{ is the common ratio}$$

For the second G.P. $\left(\frac{1}{a_n}\right)$, common ratio = $\frac{1}{r}$

$$S'_n = \frac{1}{a_1} \frac{\left(1 - \frac{1}{r^n}\right)}{\left(1 - \frac{1}{r}\right)} = \frac{(r^n - 1)}{a_1(r-1)r^{n-1}} = \frac{r^n - 1}{a_n(r-1)}$$

$$\Rightarrow S'_n = \frac{1}{a_1 a_n} + \frac{a_1(r^n - 1)}{r - 1}$$

$$\Rightarrow S'_n = \frac{1}{a_1 a_n} S_n$$

$$\Rightarrow S_n = S'_n a_1 a_n$$

Example : 19

At what values of parameter 'a' are there values of n such that the numbers : $5^{1+x} + 5^{1-x}$, $a/2$, $25^x + 25^{-x}$ form an A.P.?

Solution

For the given numbers to be in A.P.

$$2 \left(\frac{a}{2} \right) = 5^{1+x} + 5^{1-x} + 25^x + 25^{-x}$$

Let $5^x = k$

$$\Rightarrow a = 5k + \frac{5}{k} + k^2 + \frac{1}{k^2}$$

$$\Rightarrow a = 5 \left(k + \frac{1}{k} \right) + \left(k^2 + \frac{1}{k^2} \right)$$

As the sum of positive number and its reciprocal is always greater than or equal to 2,

$$k + \frac{1}{k} \geq 2 \quad \text{and} \quad k^2 + \frac{1}{k^2} \geq 2$$

$$\text{Hence } a \geq 5(2) + 2 \quad \Rightarrow \quad a \geq 12$$

Example : 20

The series of natural numbers is divided into groups : (1) : (2, 3, 4); (5, 6, 7, 8, 9) and so on.

Show that the sum of the numbers in the nth group is $(n-1)^3 + n^3$

Solution

Note that the last term of each group is the square of a natural number. Hence first term in the nth group is $(n-1)^2 + 1 = n^2 - 2n + 2$

There is 1 term in Ist group, 3 in IInd, 5 in IIIrd, 7 in IVth,

No. of terms in the nth group = nth term of (1, 3, 5, 7) = $2n - 1$

Common difference in the nth group = 1

$$\begin{aligned} \text{Sum} &= \frac{2n-1}{2} [2(n^2 - 2n + 2) + (2n - 2) \cdot 1] \\ &= \frac{2n-1}{2} [2n^2 - 2n + 2] = (2n-1)(n^2 - n + 1) \\ &= 2n^3 - 3n^2 + 3n - 1 = n^3 + (n-1)^3 \end{aligned}$$

Example : 21

If $\frac{1}{a} + \frac{1}{c} + \frac{1}{a-b} + \frac{1}{c-b} = 0$, prove that a, b, c are in H.P., unless $b = a + c$

Solution

$$\frac{1}{a} + \frac{1}{c} + \frac{1}{a-b} + \frac{1}{c-b} = 0$$

$$\Rightarrow \frac{a+c}{ac} + \frac{a+c-2b}{ac-b(a+c)+b^2} = 0$$

Let $a + c = t$

$$\Rightarrow \frac{1}{ac} + \frac{t-2b}{ac-bt+b^2} = 0$$

$$\Rightarrow act - bt^2 + b^2t + act - 2abc = 0$$

$$\Rightarrow bt^2 - b^2t - 2act + 2abc = 0$$

$$\Rightarrow bt(t-b) - 2ac(t-b) = 0$$

$$\Rightarrow (t-b)(bt-2ac) = 0$$

$$\Rightarrow t = b \quad \text{or} \quad bt = 2ac$$

$$\Rightarrow a + c = b \quad \text{or} \quad b(a + c) = 2ac$$

$$\Rightarrow a + c = b \quad \text{or} \quad b = \frac{2ac}{a + c}$$

$$\Rightarrow a, b, c \text{ are in H.P. or } a + c = b$$

Example : 22

If $a_1, a_2, a_3, \dots, a_n$ are in HP, prove that : $a_1 a_2 + a_2 a_3 + \dots + a_{n-1} a_n = (n - 1) a_1 a_n$.

Solution

Let D be the common difference of the A.P. corresponding to the given H.P.

$$\Rightarrow \frac{1}{a_n} = \frac{1}{a_1} + (n - 1) D \quad \dots\dots\dots(i)$$

Now $\frac{1}{a_1}, \frac{1}{a_2}, \frac{1}{a_3}, \dots$ are in A.p.

$$\Rightarrow \frac{1}{a_2} - \frac{1}{a_1} = D$$

$$\Rightarrow a_1 a_2 = \frac{a_1 - a_2}{D} \quad \text{and} \quad a_2 a_3 = \frac{a_2 - a_3}{D} \quad \text{and so on.}$$

$$\Rightarrow a_{n-1} a_n = \frac{a_{n-1} - a_n}{D}$$

Adding all such expressions we get

$$\Rightarrow a_1 a_2 + a_2 a_3 + a_3 a_4 + \dots + a_{n-1} a_n = \frac{a_1 - a_n}{D}$$

$$\Rightarrow a_1 a_2 + a_2 a_3 + \dots + a_{n-1} a_n = \frac{a_1 a_n}{D} \left(\frac{1}{a_n} - \frac{1}{a_1} \right)$$

$$\Rightarrow a_1 a_2 + a_2 a_3 + \dots + a_{n-1} a_n = \frac{a_1 a_n}{D} [(n - 1)D] \quad \dots\dots\dots \text{using (i)}$$

Hence $a_1 a_2 + a_2 a_3 + \dots + a_{n-1} a_n = (n - 1) a_1 a_n$

Example : 23

If p be the first of n AM's between two numbers ; q be the first of n HM's between the same numbers, prove

that the value of q cannot lie between p and $\left(\frac{n+1}{n-1}\right)^2 p$.

Solution

Let the two numbers be a and b. If p is first of n AM's then :

$$p = a + \frac{b - a}{n + 1} = \frac{b + an}{n + 1} \quad \dots\dots\dots(i)$$

If q is first of n HM s then :

$$\frac{1}{q} = \frac{1}{a} + \frac{\frac{1}{b} - \frac{1}{a}}{n + 1} \quad \Rightarrow \quad q = \frac{ab(n + 1)}{bn + a} \quad \dots\dots\dots(ii)$$

Dividing (ii) by (i) we get $\frac{q}{p} = \frac{ab(n + 1)^2}{(bn + a)(an + b)}$

$$\Rightarrow \frac{p}{q} = \frac{(n+1)^2}{n^2 + 1 + n\left(\frac{a}{b} + \frac{b}{a}\right)}$$

As the sum of a number and its reciprocal cannot lie between -2 and $+2$

$$\Rightarrow 2 \leq \frac{a}{b} + \frac{b}{a} \leq -2$$

$$\Rightarrow (n+1)^2 \leq n\left(\frac{a}{b} + \frac{b}{a}\right) + n^2 + 1 \leq (n-1)^2$$

$$\Rightarrow \frac{1}{(n+1)^2} \geq \frac{q}{p(n+1)} \geq \frac{1}{(n-1)^2}$$

$$\Rightarrow p \geq q \geq p\left(\frac{n+1}{n-1}\right)^2$$

$$\Rightarrow q \text{ cannot lie between } p \text{ and } p\left(\frac{n+1}{n-1}\right)^2$$

Example : 24

If a, b, c are in A.P., α, β, γ are in A.P. and $a\alpha, b\beta, c\gamma$ are in G.P., prove that $a : b : c = \frac{1}{\gamma} : \frac{1}{\beta} : \frac{1}{\alpha}$.

Solution

a, b, c are in A.P.(i)

$$\Rightarrow 2b = a + c$$

α, β, γ are in H.P.

$$\Rightarrow \beta = \frac{2\alpha\gamma}{\alpha + \gamma} \dots\dots(ii)$$

$a\alpha, b\beta, c\gamma$ are in GP

$$\Rightarrow b^2\beta^2 = a\alpha c\gamma \dots\dots(iii)$$

Using (i), (ii) and (iii), $\left(\frac{a+c}{2}\right)^2 \left(\frac{2\alpha\gamma}{\alpha+\gamma}\right)^2 = a\alpha c\gamma$

$$\frac{(a+c)^2}{ac} = \frac{(\alpha+\gamma)^2}{\alpha\gamma}$$

$$\Rightarrow \frac{a}{c} + \frac{c}{a} = \frac{\alpha}{\gamma} + \frac{\gamma}{\alpha}$$

Multiplying by $\frac{\alpha}{\gamma}$ we get,

$$\frac{\alpha^2}{\gamma^2} - \frac{\alpha}{\gamma} \left(\frac{a}{c} + \frac{c}{a}\right) + 1 = 0$$

$$\Rightarrow \frac{\alpha}{\gamma} \left(\frac{\alpha}{\gamma} - \frac{a}{c}\right) - \frac{c}{a} \left(\frac{\alpha}{\gamma} - \frac{a}{c}\right) = 0$$

$$\Rightarrow \left(\frac{\alpha}{\gamma} - \frac{c}{a}\right) \left(\frac{\alpha}{\gamma} - \frac{a}{c}\right) = 0$$

$$\Rightarrow a\alpha = c\gamma \quad \text{or} \quad c\alpha = a\gamma$$

$a\alpha = c\gamma$ is not possible as $a\alpha, b\beta, c\gamma$ are in GP {obviously with common ratio $\neq 1$ }
Hence we have only $c\alpha = a\gamma$
Using this in (iii)
 $b^2\beta^2 = a2\gamma^2 \Rightarrow b\gamma = a\gamma$

$$\Rightarrow \frac{1}{1/\gamma} = \frac{b}{1/\beta} = \frac{c}{1/\alpha}$$

$$\Rightarrow a : b : c = \frac{1}{\gamma} : \frac{1}{\beta} : \frac{1}{\alpha}$$

Example : 25

Find three numbers a, b, c between 2 and 18 such that :

- (i) their sum is 25,
- (ii) the numbers 2, a, b are consecutive terms of an A.P.
- (iii) the numbers b, c, 18 are consecutive terms of a G.P.

Solution

According to the given condition, we have

$$a + b + c = 25 \quad \dots\dots\dots(i)$$

$$2a = b + 2 \quad \dots\dots\dots(ii)$$

$$c^2 = 18b \quad \dots\dots\dots(iii)$$

eliminating b and a, using (ii) and (iii) we get

$$\frac{1}{2} \left(\frac{c^2}{18} + 2 \right) + \frac{c^2}{18} + c = 25$$

$$\Rightarrow c^2 + 36 + 2c^2 + 36c = 25 \quad (36)$$

$$\Rightarrow (c + 24)(c - 12) = 0$$

$$\Rightarrow c = 12, -24$$

As a, b, c are between 2 and 18, c = 12 is the only solution

Using (iii), $b = c^2/18 = 8$

Using (ii), $a = \frac{b+2}{2} = 5$

Hence a = 5, b = 8, c = 12

Example : 26

If a, b, c are in G.P., and the equations $ax^2 + 2bx + c = 0$ and $dx^2 + 2ex + f = 0$ have a common root then show that $d/a, e/b, f/c$ are in A.P.

Solution

a, b, c are in G.P. $\Rightarrow b^2 = ac$

Hence the first equation has real roots because its discriminant = $4b^2 - 4ac = 0$

the roots are $x = \frac{-2b}{2a} = -\frac{b}{a}$

As the two equations have a common roots, $-b/a$ is root of the second equation also.

$$\Rightarrow d \left(-\frac{b}{a} \right)^2 + 2e \left(-\frac{b}{a} \right) + f = 0$$

$$\Rightarrow db^2 - 2abc + a^2f = 0$$

dividing by $ab^2 \Rightarrow \frac{d}{a} - \frac{2e}{b} + \frac{a^2f}{ab^2} = 0$

$$\Rightarrow \frac{d}{a} - \frac{2e}{b} + \frac{a^2f}{a(ac)} = 0 \Rightarrow \frac{d}{a} - \frac{2e}{b} + \frac{f}{c} = 0$$

$$\Rightarrow \frac{d}{a}, \frac{e}{b}, \frac{f}{c} \text{ are in A.P.}$$

Example : 27

The sum of the squares of three distinct real numbers, which are in G.P. is S^2 . If their sum is αS , show that

$$: \alpha^2 \in \left(\frac{1}{3}, 1\right) \cup (1, 3)$$

Solution

Let the numbers be b, br and br^2

$$b^2 + b^2 r^2 + b^2 r^4 = S^2$$

$$b + br + br^2 = \alpha S$$

eliminating S , we get

$$\frac{b^2(1+r^2+r^4)}{b^2(1+r+r^2)^2} = \frac{S^2}{\alpha^2 S^2}$$

$$\Rightarrow \alpha^2 = \frac{(1+r+r^2)^2}{1+r^2+r^4} = \frac{(1+r+r^2)^2}{(1+r^2)^2 - r^2}$$

$$\Rightarrow \alpha^2 = \frac{(1+r+r^2)^2}{(1+r+r^2)(1+r^2-r)} = \frac{1+r+r^2}{1-r+r^2}$$

$$\Rightarrow r^2(\alpha^2 - 1) - r(\alpha^2 + 1) + \alpha^2 - 1 = 0$$

as r is real, this quadratic must have non-negative discriminant

$$\Rightarrow (\alpha^2 + 1)^2 - 4(\alpha^2 - 1)(\alpha^2 - 1) \geq 0$$

$$\Rightarrow [\alpha^2 + 1 + 2(\alpha^2 - 1)][\alpha^2 + 1 - 2(\alpha^2 - 1)] \geq 0$$

$$\Rightarrow (3\alpha^2 - 1)(3 - \alpha^2) \geq 0$$

$$\Rightarrow (\alpha^2 - 1/3)(\alpha^2 - 3) \leq 0$$

As the numbers in G.P. are distinct, the following cases should be excluded.

$$\alpha^2 = 3 \quad \Rightarrow \quad r = 1$$

$$\alpha^2 = 1/3 \quad \Rightarrow \quad r = 1$$

$$\alpha^2 = 1 \quad \Rightarrow \quad r = 0$$

Hence α^2 is between $1/3$ and 3 , but not equal to 1 .

$$\Rightarrow \alpha^2 \in \left(\frac{1}{3}, 1\right) \cup (1, 3)$$

Example : 28

If the first and the $(2n - 1)$ st term of an A.P., a G.P. and a H.P. are equal and their n th terms are a, b and c respectively, then :

$$(A) \quad a = b = c \quad (B) \quad a \geq b \geq c \quad (C) \quad a + c = b \quad (D) \quad ac - b^2 = 0$$

Solution

Let the first term = A

The last term $[(2n - 1)$ st term] = L

No. of terms $2n - 1$ i.e. odd

$$\text{Middle term} = \frac{(2n-1)+1}{2} = n^{\text{th}} \text{ term}$$

$\Rightarrow T_n$ is the middle term for all the three progressions. In an A.P. the middle term is the arithmetic mean of first and the last terms.

$$\Rightarrow a = \frac{A+L}{2}$$

In a G.P. the middle term is the geometric mean of first and last terms.

$$\Rightarrow b = \sqrt{AL}$$

In an H.P. the middle term is the harmonic mean of first and last terms.

$$\Rightarrow c = \frac{2AL}{A+L}$$

Hence a, b, c are AM, GM and HM between the numbers A and L.

As $(GM)^2 = (AM)(HM)$

We have $b^2 = ac$

\Rightarrow (B) and (D) are the correct choices.

Example : 29

Sum of the series : $1 + 3x + 5x^2 + 7x^2 + \dots$

Solution

Note that the given series is an Arithmetico-Geometric series.

$1, 3, 5, \dots$ are in A.P. $\Rightarrow T_n = 2n - 1$

$1, x, x^2, \dots$ are in G.P. $\Rightarrow T_n = x^{n-1}$

(a) This means that n^{th} term of A - G series = $(2n - 1) x^{n-1}$

$$S = 1 + 3x + 5x^2 + \dots + (2n - 3) x^{n-2} + (2n - 1) x^{n-1} \dots (i)$$

$$xS = x + 3x^2 + 5x^3 + \dots + (2n - 3) x^{n-1} + (2n - 1)x^n \dots (ii)$$

$$\Rightarrow (1 - x) S = 1 + 2x + 2x^2 + \dots + 2x^{n-1} - (2n - 1) x^n$$

$$\Rightarrow (1 - x) S = 1 + \frac{2x(1 - x^{n-1})}{1 - x} - (2n - 1) x^n$$

$$\Rightarrow S = \frac{1}{1 - x} + \frac{2x(1 - x^{n-1})}{(1 - x)^2} - \frac{(2n - 1)x^n}{1 - x}$$

(b) $S_n = 1 + 3x + 5x^2 + \dots$ to ∞

$xS_n = x + 3x^2 + 5x^3 + \dots$ to ∞

$$\Rightarrow (1 - x) S_\infty = 1 + 2x + 2x^2 + \dots$$
 to ∞

$$\Rightarrow (1 - x) S_\infty = 1 + 2x (1 + x + x^2 + \dots)$$
 to ∞

$$\Rightarrow (1 - x) S_\infty = 1 + 2x \left(\frac{1}{1 - x} \right) = \frac{1 + x}{1 - x}$$

$$\Rightarrow S_\infty = \frac{1 + x}{(1 - x)^2}$$

Example : 30

Sum the series : $1.2.3 + 2.3.4 + 3.4.5 + \dots +$ to n terms.

Solution

Here $T_n = n(n + 1)(n + 2)$

$$\Rightarrow T_n = n^3 + 3n^2 + 2n$$

$$\Rightarrow T_n = n^3 + 3n^2 + 2n$$

$$\Rightarrow S_n = \sum T_n = \sum n^3 + 3 \sum n^2 + 2 \sum n$$

$$\Rightarrow \frac{n^2(n+1)^2}{4} + \frac{2n(n+1)(2n+1)}{6} + \frac{2n(n+1)}{2}$$

$$\Rightarrow \frac{n(n+1)}{4} [n(n+1) + 2(2n+1) + 4]$$

$$\Rightarrow \frac{n(n+1)}{4} [n^2 + 5n + 6] = \frac{n(n+1)(n+2)(n+3)}{4}$$

Example : 31

Sum the series : $1^2 + (1^2 + 2^2) + (1^2 + 2^2 + 3^2) + \dots$ to n terms.

Solution

First determine the nth term.

$$\Rightarrow T_n = (1^2 + 2^2 + 3^2 + \dots + n^2)$$

$$\Rightarrow T_n = \sum n^2 = \frac{n(n+1)(2n+1)}{6}$$

$$\Rightarrow T_n = \frac{1}{3} n^3 + \frac{1}{2} n^2 + \frac{1}{6} n$$

$$\text{Now } S_n = \sum T_n = \frac{1}{3} \sum n^3 + \frac{1}{2} \sum n^2 + \frac{1}{6} \sum n$$

$$S_n = \frac{1}{3} \frac{n^2(n+1)^2}{4} + \frac{1}{2} \frac{n(n+1)(2n+1)}{6} + \frac{1}{6} \frac{n(n+1)}{2}$$

$$\text{Simplify to get } S_n = \frac{n(n+1)^2(n+2)}{12}$$

Example : 32

Find the sum of n terms of series : $(x + y) + (x^2 + xy + y^2) + (x^3 + x^2y + xy^2 + y^3) + \dots$

Solution

$$\text{Let } S_n = (x + y) + (x^2 + xy + y^2) + (x^3 + x^2y + xy^2 + y^3) + \dots$$

$$S_n = \frac{x^2 - y^2}{x - y} + \frac{x^3 - y^3}{x - y} + \frac{x^4 - y^4}{x - y} + \dots$$

$$= \frac{1}{x - y} (x^2 + x^3 + x^4 + \dots) - \frac{1}{x - y} (y^2 + y^3 + y^4 + \dots)$$

$$= \frac{1}{x - y} \left(\frac{x^2(1 - x^n)}{1 - x} \right) - \frac{1}{x - y} \left(\frac{y^2(1 - y^n)}{1 - y} \right)$$

Note : The following results can be very useful

$$(i) \quad \frac{x^n - y^n}{x - y} = x^{n-1} + x^{n-2} + x^{n-3} + \dots + xy^{n-2} + y^{n-1} \quad (n \text{ is an integer})$$

$$(ii) \quad \frac{x^n + y^n}{x + y} = x^{n-1} - x^{n-2} + x^{n-3} y^2 - x^{n-4} y^3 + \dots + y^{n-1} \quad (n \text{ is odd})$$

Example : 33

Sum the series : $1 + \frac{4}{5} + \frac{7}{5^2} + \frac{10}{5^3} + \dots$ to n terms and to ∞ .

Solution

The given series is arithmetico-geometric series

$$\text{Let } S = 1 + \frac{4}{5} + \frac{7}{5^2} + \dots + \frac{3n-2}{5^{n-1}}$$

$$\frac{1}{5} S = \frac{1}{5} + \frac{4}{5^2} + \dots + \frac{3n-5}{5^{n-1}} + \frac{3n-2}{5^n}$$

$$\Rightarrow \frac{4}{5} S = 1 + \left(\frac{3}{5} + \frac{3}{5^2} + \dots + \frac{3}{5^{n-1}} \right) - \left(\frac{3n-2}{5^n} \right)$$

$$\Rightarrow S = \frac{5}{4} + \frac{5}{4} \times \frac{3}{5} \left(\frac{1 - \frac{1}{5^{n-1}}}{1 - \frac{1}{5}} \right) - \left(\frac{3n-2}{5^n} \times \frac{5}{4} \right)$$

$$\Rightarrow S = \frac{5}{4} + \frac{3}{4} \left(\frac{5^{n-1}-1}{4} \right) \frac{1}{5^{n-2}} - \frac{3n-2}{4 \cdot 5^{n-1}}$$

$$\Rightarrow S = \frac{5}{4} + \frac{15}{16} - \frac{3}{16 \cdot 5^{n-2}} - \frac{3n-2}{20 \cdot 5^{n-2}}$$

$$\Rightarrow S = \frac{35}{16} - \left(\frac{12n+7}{80(5^{n-2})} \right)$$

Now $S_n = 1 + \frac{4}{5} = \frac{7}{5^2} + \dots \dots \dots \infty$

$$\frac{1}{5} S_\infty = \frac{1}{5} + \frac{4}{5^2} + \dots \dots \dots \infty$$

$$\Rightarrow \frac{4}{5} S_\infty = 1 + \frac{3}{5} + \frac{3}{5^2} + \dots \dots \dots \infty$$

$$\Rightarrow \frac{4}{5} S_\infty = 1 + \frac{3/5}{1-1/5}$$

$$\Rightarrow S_\infty = \frac{5}{4} \left(1 + \frac{3}{4} \right) = \frac{35}{16}$$

$$\frac{(n+1)^2}{4}$$

Example : 34

Sum the series :

$$\frac{1^3}{1} + \frac{1^3+2^3}{1+3} + \frac{1^3+2^3+3^3}{1+3+5} + \dots \dots \dots \text{to } n \text{ terms.}$$

Solution

$$T_n = \frac{1^3 + 2^3 + \dots + n^2}{1 + 3 + 5 + \dots + n}$$

$$T_n = \frac{\sum n^3}{\frac{n}{2}[2 + (n-1)2]} =$$

$$\Rightarrow T_n = \frac{n^2 + 2n + 1}{4}$$

$$S_n = \sum T_n = \frac{1}{4} \left[\sum n^2 + 2 \sum n + \sum 1 \right] = \frac{1}{4} \left[\frac{n(n+1)(2n+1)}{6} + n(n-1) + n \right] = \frac{n}{24} [2n^2 + 3n + 1 + 6n + 6 + 6]$$

$$\Rightarrow S_n = \frac{n}{24} [2n^2 + 9n + 12]$$

Example : 35

Find the sum of the products of every pair of the first n natural numbers.

Solution

The required sum is given as follows.

$$S = 1.2 + 1.3 + 1.4 + \dots + 2.3 + 2.4 + \dots + 3.4 + 3.5 + \dots + \dots + (n-1)n$$

Using : $S = \frac{(\sum n)^2 - \sum n^2}{2}$, we get :

$$S = \frac{1}{2} \left[\frac{n^2(n+1)^2}{4} - \frac{n(n+1)(2n+1)}{6} \right] = \frac{n(n+1)}{24} [3n(n+1) - 2(2n+1)]$$

$$= \frac{n(n+1)}{24} [3n^2 - n - 2]$$

$$\Rightarrow S_n = \frac{n(n+1)(3n+2)(n-1)}{24}$$

Example : 36

Find the sum of first n terms of the series : 3 + 7 + 13 + 21 + 31 +

Solution

The given series is neither an A.P. nor a G.P. but the difference of the successive terms are in A.P.

Series : 3 7 13 21 31

Differences : 4 6 8 10

In such cases, we find the nth term as follows :

Let S be the sum of the first n terms.

$$S = 3 + 7 + 13 + 21 + 31 + \dots + T_n$$

$$S = 3 + 7 + 13 + 21 + 31 + \dots + T_{n-1} + T_n$$

On subtracting, we get :

$$\Rightarrow 0 = 3 + \{4 + 6 + 8 + 10 + \dots\} - T_n$$

$$\Rightarrow T_n = 3 + \{4 + 6 + 8 + 10 + \dots(n-1) \text{ terms}\}$$

$$\Rightarrow T_n = 3 + \frac{n-1}{2} [2(4) + (n-2) 2]$$

$$\Rightarrow T_n = n^2 + n + 1$$

$$\Rightarrow S = \sum_{k=1}^n T_k = \sum k^2 + \sum k + \sum 1$$

$$\Rightarrow \frac{n(n+1)(2n+1)}{6} + \frac{n(n+1)}{2} + n = \frac{n}{3} (n^2 + 3n + 5)$$

Example : 37

Sum the series : 1 + 4 + 10 + 22 + 46 + to n terms.

Solution

The differences of successive terms are in G.P. :

Series : 1 4 10 22 46

Differences : 3 6 12 24

Let S = sum of first n terms.

$$\Rightarrow S = 1 + 4 + 10 + 22 + 46 + \dots + T_n$$

$$\Rightarrow S = 1 + 4 + 10 + 22 + 46 + \dots + T_{n-1} + T_n$$

On subtracting, we get

$$0 = 1 + \{3 + 6 + 12 + 24 + \dots\} - T_n$$

$$T_n = 1 + \{3 + 6 + 12 + 24 + \dots(n-1) \text{ terms}\}$$

$$\Rightarrow T_n = 1 + \frac{3(2^{n-1} - 1)}{2 - 1}$$

$$\Rightarrow T_n = 3 \cdot 2^{n-1} - 2$$

$$\begin{aligned} \Rightarrow S &= \sum_{k=1}^n T_k = \frac{3}{2} \sum 2^k - \sum 2 = \frac{3}{2} (2 + 4 + 8 + \dots + 2n) - 2n \\ &= \frac{3}{2} \frac{2(2^n - 1)}{2 - 1} - 2n = 3 \cdot 2^n - 2^n - 3 \end{aligned}$$

Example : 38

Find the sum of the series : $\frac{1}{1.4} + \frac{1}{4.7} + \frac{1}{7.10} + \dots$ to n terms

Solution

$$\text{Let } S = \frac{1}{1.4} + \frac{1}{4.7} + \frac{1}{7.10} + \dots + \frac{1}{(3n-2)(3n+1)}$$

$$\Rightarrow 3S = \frac{3}{1.4} + \frac{3}{4.7} + \frac{3}{7.10} + \dots + \frac{3}{(3n-2)(3n+1)}$$

$$\Rightarrow 3S = \frac{4-1}{1.4} + \frac{7-4}{4.7} + \frac{10-7}{7.10} + \dots + \frac{(3n+1)(3n-2)}{(3n-2)(3n+1)}$$

$$\Rightarrow 3S = \left(\frac{1}{1} - \frac{1}{4}\right) + \left(\frac{1}{4} - \frac{1}{7}\right) + \left(\frac{1}{7} - \frac{1}{10}\right) + \dots + \left(\frac{1}{(3n-2)} - \frac{1}{3n+1}\right)$$

$$\Rightarrow S = \frac{1}{1} - \frac{1}{3n+1}$$

$$\Rightarrow S = \frac{n}{3n+1}$$

Note : The above method works in the case when nth term of a series can be expressed as the difference of the two quantities of the type :

$$T_n = f(n) - f(n-1)$$

or

$$T_n = f(n) - f(n+1)$$

$$\text{In the above example, } T_n = \frac{1}{(3n-2)(3n+1)} = \frac{1}{3} \left(\frac{1}{3n-2} - \frac{1}{3n+1} \right)$$

It is in the form $f(n) - f(n+1)$

Example : 39

Find the sum of first n terms of the series : $\frac{1}{1.2.3} + \frac{1}{2.3.4} + \frac{1}{3.4.5} + \frac{1}{4.5.6} + \dots$

Solution

$$\text{Let } S = \frac{1}{1.2.3} + \frac{1}{2.3.4} + \frac{1}{3.4.5} + \dots + \frac{1}{n(n+1)(n+2)}$$

$$2S = \frac{3-1}{1.2.3} + \frac{4-2}{2.3.4} + \frac{5-3}{3.4.5} + \dots + \frac{(n+2)-n}{n(n+1)(n+2)}$$

$$2S = \left(\frac{1}{1.2} - \frac{1}{2.3}\right) + \left(\frac{1}{2.3} - \frac{1}{3.4}\right) + \dots + \left(\frac{1}{n(n+1)} - \frac{1}{(n+1)(n+2)}\right)$$

$$2S = \frac{1}{1.2} - \frac{1}{(n+1)(n+2)}$$

$$\Rightarrow S = \frac{1}{4} - \frac{1}{2(n-1)(n+2)}$$

Note : You should observe that here,

$$T_n = \frac{1}{n(n+1)(n+2)} = \frac{1}{2} \left(\frac{1}{n(n+1)} - \frac{1}{(n+1)(n+2)} \right)$$

It is in the form $f(n) - f(n+1)$

Example : 40

Find the sum of first n terms of the series : $1(1)! + 2(2)! + 3(3)! + 4(4)! + \dots$

Solution

The nth term = $T_n = n(n)!$

T_n can be written as

$$\begin{aligned} T_n &= (n+1-1)(n)! \\ \Rightarrow T_n &= (n+1)! - (n)! \quad \dots\dots\dots(i) \end{aligned}$$

This is in the form $f(n) - f(n-1)$

$$\begin{aligned} S &= T_1 + T_2 + T_3 + T_4 + \dots\dots\dots + T_n \\ S &= (2! - 1!) + (3! - 2!) + (4! - 3!) + \dots\dots\dots + \{(n+1)! - n!\} \\ \Rightarrow S &= -1! + (n+1)! \\ \Rightarrow S &= (n+1)! - 1 \end{aligned}$$

Example : 41

Sum the series to n terms : $4 + 44 + 444 + 4444 + \dots$ to n terms.

Solution

$$\begin{aligned} \text{Let } S_n &= 4 + 44 + 444 + 4444 + \dots\dots\dots \text{ to n terms} \\ \Rightarrow S_n &= 4(1 + 11 + 111 + 1111 + \dots\dots\dots \text{ n terms}) \\ \Rightarrow S_n &= \frac{4}{9} \{(10 - 1) + (100 - 1) + (1000 - 1) + \dots\dots\dots \text{ n terms}\} \\ \Rightarrow S_n &= \frac{4}{9} \{(10 + 10^2 + 10^3 + \dots\dots\dots \text{ n terms}) - (1 + 1 + 1 + \dots\dots\dots \text{ n terms})\} \end{aligned}$$

$$S_n = \frac{4}{9} \left(\frac{10(10^n - 1)}{10 - 1} - n \right)$$

$$\Rightarrow S_n = \frac{4}{81} [10(10^n - 1) - 9n]$$