Example : 1

What can you say about the roots of the following equations ?

(i) $x^2 + 2(3a + 5) x + 2 (9a^2 + 25) = 0$

(ii) $(y – a) (y – b) + (y – b) (y – c) + (y – c) (y – a) = 0$

Solution :

(i) Calculate Discriminant D

$D = 4(3a + 5^2) – 8(9a^2 + 25)$

$D = – 4(3a – 5)^2$

$\Rightarrow$ $D \le 0$, so the roots are :

complex if $a \neq 5/3$ and real and equal if $a = 5/3$.

(ii) Simplifying the given equation ;

$3y^2 – 2(a + b + c) y + (ab + bc + ca) = 0$

$\Rightarrow$ $D = 4(a + b + c)^2 – 12 (ab + bc + ca)$

$\Rightarrow$ $D = 4(a^2 + b^2 + c^2 – ab – bc – ca)$

Now using the identity

$$(a^2 + b^2 + c^2 – ab – bc – ca) = \frac{1}{2} [(a – b)^2 + (b – c)^2 + (c – a)^2]$$

we get :

$D = 2[(a – b)^2 + (b – c)^2 + (c – a)^2]$

$\Rightarrow$ $D \ge 0$, so the roots are real

Note : if D = 0, then $(a – b)^2 + (b – c)^2 + (c – a)^2 = 0$

$\Rightarrow$ $a = b = c$

$\Rightarrow$ if $a = b = c$, then the root are equal

Example : 2

Find the value of k, so that the equations $2x^2 + kx – 5 = 0$ and $x^2 – 3x – 4 = 0$ may have one root in common.

Solution :

Let α be common root of two equations.

Hence $2\alpha^2 + k\alpha – 5 = 0$ and $\alpha^2 – 3\alpha – 4 = 0$

Solving the two equations;

$$\frac{\alpha^2}{-4k-15} = \frac{-\alpha}{-8+5} = \frac{1}{-6-k}$$

$\Rightarrow$ $(–3)^2 = (4k + 15) (6 + k)$

$\Rightarrow$ $4k^2 + 39k + 81 = 0$

$\Rightarrow$ $k = – 3$ or $k = –27/4$

Example : 3

If $ax^2 + bx + c = 0$ and $bx^2 + cx + a = 0$ have a root in common, find the relation between a, b and c.

Solution

Solve the two equations as done in last example,

$ax^2 + bx + c = 0$ and $bx^2 + cx + a = 0$

$$\frac{x^2}{ba-c^2} = \frac{-x}{a^2-bc} = \frac{1}{ac-b^2}$$

$\Rightarrow$ $(a^2 – bc)^2 = (ba – c^2) (ac – b^2)$

simplifying to get : $a (a^3 + b^3 + c^3 – 3abc) = 0$

$\Rightarrow$ $a = 0$ or $a^3 + b^3 + c^3 = 3abc$

This is the relation between a, b and c.

Example : 4

If α, β are the roots of $x^2 + px + q = 0$ and γ, δ are the roots of $x^2 + rx + s = 0$, evaluate the value of $(\alpha - \gamma)(\alpha - \delta)(\beta - \gamma)(\beta - \delta)$ in terms of p, q, r, s. Hence deduce the condition that the equation have a common root.

Solution

Let α, β be the roots of $x^2 + px + q = 0$

$\Rightarrow \quad \alpha + \beta = -p \quad$ and $\quad \alpha\beta = q \quad$ ....(i)

γ, δ be the roots of $x^2 + rx + s = 0$

$\Rightarrow \quad \gamma + \delta = -r \quad$ and $\quad \gamma\delta = s \quad$ ....(ii)

Expanding $\quad (\alpha - \gamma)(\alpha - \delta)(\beta - \gamma)(\beta - \delta)$

$= [\alpha^2 - (\gamma + \delta)\alpha + \gamma\delta]\ [\beta^2 - (\gamma + \delta)\beta + \gamma\delta]$

...............[using (i) and (ii)]

$= (\alpha^2 - r\alpha + s)(\beta^2 + r\beta + s)$

As α is a root of $x^2 + px + q = 0$

we have $\alpha^2 + p\alpha + q = 0$

and similarly $\beta^2 + p\beta + q = 0$

Substituting the values of α^2 and β^2 , and we get;

$(\alpha - \gamma)(\alpha - \delta)(\beta - \gamma)(\beta - \delta)$

$= (-p\alpha - q + r\alpha + s)(-p\beta - q + r\beta + s)$

$= [(r - p)\alpha + s - q]\ [(r - p)\beta + s - q]$

$= (r - p)^2\alpha\beta + (s - q)^2 + (s - q)(r - p)(\alpha + \beta)$

$= (r - p)^2 q + (s - q)^2 - p(s - q)(r - p)$

$= (r - p)(rq - pq - ps + pq) + (s - q)^2$

$= (r - p)(qr - ps) + (s - q)^2$

If the equation have a common root then either

$\alpha = \gamma$ or $\alpha = \delta$ or $\beta = \gamma$ or $\beta = \delta$

i.e. $\quad (\alpha - \gamma)(\alpha - \delta)(\beta - \gamma)(\beta - \delta) = 0$

$\Rightarrow \quad (s - q)^2 + (r - p)(qr - ps) = 0$

$\Rightarrow \quad (s - q)^2 = (r - p)(ps - qr)$

Example : 5

If the ratio of roots of the equation $x^2 + px + q = 0$ be equal to the ratio of roots of the equation $x^2 + bx + c = 0$, then prove that $p^2c = b^2q$.

Solution

Let α and β be the roots of $x^2 + px + q = 0$ and γ, δ be the roots of equation $x^2 + bx + c = 0$

$$\Rightarrow \quad \frac{\alpha}{\beta} = \frac{\gamma}{\delta} \qquad \Rightarrow \quad \frac{\alpha}{\gamma} = \frac{\beta}{\delta}$$

$$\Rightarrow \quad \frac{\alpha}{\gamma} = \frac{\beta}{\delta} = \frac{\alpha+\beta}{\gamma+\delta} = \frac{\sqrt{\alpha\beta}}{\sqrt{\gamma\delta}} \qquad \Rightarrow \quad \frac{\alpha+\beta}{\gamma+\delta} = \frac{\sqrt{\alpha\beta}}{\sqrt{\gamma\delta}}$$

$$\Rightarrow \quad \frac{-p}{-b} = \frac{\sqrt{q}}{\sqrt{c}} \qquad \Rightarrow \quad p^2c = b^2q$$

Another Method :

$$\frac{\alpha}{\beta} = \frac{\gamma}{\delta} \qquad \Rightarrow \quad \frac{(\alpha+\beta)^2}{(\alpha-\beta)^2} = \frac{(\gamma+\delta)^2}{(\gamma-\delta)^2}$$

$$\Rightarrow \quad \frac{(\alpha+\beta)^2}{(\alpha+\beta)^2-(\alpha-\beta)^2} = \frac{(\gamma+\delta)^2}{(\gamma+\delta)^2-(\gamma-\delta)^2} \qquad \Rightarrow \quad \frac{(\alpha+\beta)^2}{4\alpha\beta} = \frac{(\gamma+\delta)^2}{4\gamma\delta}$$

$$\Rightarrow \quad \frac{p^2}{4q} = \frac{b^2}{4c} \qquad \Rightarrow \quad p^2c = b^2q$$

Example : 6

If α is a root of $4x^2 + 2x - 1 = 0$, prove that $4\alpha^3 - 3\alpha$ is the other root.

Solution

If α is one root, then the sum of root = – 2/4 = – 1/2

$\Rightarrow$ other root = $\beta = -1/2 - \alpha$

Now we will try to prove that :

$-1/2 - \alpha$ is equal to $4\alpha^3 - 3\alpha$.

We have $4\alpha^2 + 2\alpha - 1 = 0$, because α is a root of $4x^2 + 2x - 1 = 0$

Now $4\alpha^3 - 3\alpha = \alpha\,(4\alpha^2 + 2\alpha - 1) - 2\alpha^2 - 2\alpha$

$= \alpha\,(0) - 1/2\,(4\alpha^2 + 2\alpha - 1) - 1/2 - \alpha$

$= \alpha\,(0) - 1/2\,(0) - 1/2 - \alpha = -1/2 - \alpha$

hence $4\alpha^3 - 3\alpha$ is the other root.

Example : 7

Find all the roots of the equation : $4x^4 - 24x^3 + 57x^2 + 18x - 45 = 0$ if one root is $3 + i\sqrt{6}$.

Solution

As the coefficients are real, complex roots will occur in conjugate pairs. Hence another root is $3 - i\sqrt{6}$

Let α, β be the remaining roots.

$\Rightarrow$ the four roots are $3 \pm i\sqrt{6}$, α, β

$\Rightarrow$ the factors

$= (x - 3 - i\sqrt{6}\,)\,(x - 3 + i\sqrt{6}\,)\,(x - \alpha)\,(x - \beta)$

$= [(x - 3)^2 + 6]\,(x - \alpha)\,(x - \beta)$

$= (x^2 - 6x + 15)\,(x - \alpha)\,(x - \beta)$

Dividing $4x^4 - 24x^3 + 57x^2 + 18x - 45$ by $x^2 - 6x + 15$ or by inspection we can find the other factor of quadratic equation is $4x^2 - 3$

$\Rightarrow$ $4x^4 - 24x^3 + 57x^2 + 18x - 45 = (x^2 - 6x + 15)\,(4x^2 - 3)$

$\Rightarrow$ α, β are roots of $4x^2 - 3 = 0$

$\Rightarrow$ $\alpha, \beta = \pm\sqrt{3/2}$

Hence roots are $3 \pm i\sqrt{6}$, $\pm\sqrt{3/2}$

Example : 8

Show that f(x) can never lie between 5 and 9 if $x \in R$, where : $f(x) = \dfrac{x^2 + 34x - 71}{x^2 + 2x - 7}$

Solution

Let $\dfrac{x^2 + 34x - 71}{x^2 + 2x - 7} = k$

$\Rightarrow$ $x^2\,(1 - k) + (34 - 2k)\,x + 7k - 71 = 0$

As $x \in R$, discriminant ≥ 0

$\Rightarrow$ $(34 - 2k)^2 - 4(1 - k)\,(7k - 71) \geq 0$

$\Rightarrow$ $(17 - k)^2 - (1 - k)\,(7k - 71) \geq 0$

$\Rightarrow$ $8k^2 - 112k + 360 \geq 0$

$\Rightarrow$ $k^2 - 14k + 45 \geq 0$

$\Rightarrow$ $(k - 5)\,(k - 9) \geq 0$

$\Rightarrow$ $k \in (-\infty, 5] \cup [9, \infty)$

Hence k can never lie between 5 and 9

Example : 9

Find the values of m for which the expression : $\frac{2x^2-5x+3}{4x-m}$ can take all real values for $x \in R$.

Solution

Let $\frac{2x^2-5x+3}{4x-m}$ = k

$\Rightarrow$ $2x^2 - (4k + 5)x + 3 + mk = 0$

$\Rightarrow$ as $x \in R$, discriminant ≥ 0

$\Rightarrow$ $(4k + 5)^2 - 8(3 + mk) \geq 0$

$\Rightarrow$ $16k^2 + (40 - 8m)k + 1 \geq 0$

k can take values which satisfy this inequality. Hence k will take all real values if this inequality is true for all values of k.

A quadratic in k is positive for all values of k if coefficient of k^2 is positive and discriminant ≤ 0

$\Rightarrow$ $(40 - 8m)^2 - 4(16)(1) \leq 0$

$\Rightarrow$ $(5 - m)^2 - 1 \leq 0$

$\Rightarrow$ $(m - 5 - 1)(m - 5 + 1) \leq 0$

$\Rightarrow$ $(m - 6)(m - 4) \leq 0$

$\Rightarrow$ $m \in [4, 6]$

So for the given expression to take all real values, m should take values : $m \in [4, 6]$

Example : 10

Solve for x : $\frac{8x^2+16x-51}{(2x-3)(x+4)} > 3$

Solution

$$\frac{8x^2+16x-51}{(2x-3)(x+4)} - 3 > 0$$

$$\Rightarrow \quad \frac{8x^2+16x-51-3(2x-3)(x+4)}{(2x-3)(x+4)} > 0$$

$$\Rightarrow \quad \frac{2x^2+x-15}{(2x-3)(x+4)} > 0$$

$$\Rightarrow \quad \frac{(2x-5)(x+3)}{(2x-3)(x+4)} > 0$$

Critical points are : $x = -4, -3, 3/2, 5/2$

The solution from the number line is :

$$x \in (-\infty, -4) \cup \left(-3, \frac{3}{2}\right) \cup \left(\frac{5}{2}, \infty\right)$$

Example : 11

Find the values of m so that the inequality : $\left|\frac{x^2+mx+1}{x^2+x+1}\right| < 3$ holds for all $x \in R$.

Solution

We know that $|a| < b \quad \Rightarrow \quad -b < a < b$

Hence $\left|\frac{x^2+mx+1}{x^2+x+1}\right| < 3$

$$\Rightarrow \quad -3 < \frac{x^2+mx+1}{x^2+x+1} < 3$$

First consider $\frac{x^2 + mx + 1}{x^2 + x + 1} < 3$

$$\Rightarrow \qquad \frac{(x^2 + mx + 1) - 3(x^2 + x + 1)}{x^2 + x + 1} < 0$$

$$\Rightarrow \qquad \frac{-2x^2 + (m-3)x - 2}{\left(x + \frac{1}{2}\right)^2 + \frac{3}{4}} < 0$$

multiplying both sides by denominator, we get :

$\Rightarrow$ $-2x^2 + (m-3)x - 2 < 0$

(because denominator is always positive)

$\Rightarrow$ $2x^2 - (m-3)x + 2 > 0$

A quadratic expression in x is always positive if :

coefficient of $x^2 > 0$ and $D < 0$

$\Rightarrow$ $(m-3)^2 - 4(2)(2) < 0$

$\Rightarrow$ $m^2 - 6m - 7 < 0$

$\Rightarrow$ $(m-7)(m+1) < 0$

$\Rightarrow$ $m \in (-1, 7)$ ....(i)

Now consider $-3 < \frac{x^2 + mx + 1}{x^2 + x + 1}$

$$\Rightarrow \qquad \frac{(x^2 + mx + 1) + 3(x^2 + x + 1)}{\left(x + \frac{1}{2}\right)^2 + \frac{3}{4}} > 0$$

$\Rightarrow$ $4x^2 + (m+3)x + 4 > 0$ $\qquad \frac{1}{x+1}$

For this to be true, for all $x \in R$, $D < 0$

$\Rightarrow$ $(m+3)^2 - 4(4)(4) < 0$

$\Rightarrow$ $m^2 + 6m - 55 < 0$

$\Rightarrow$ $(m-5)(m+11) < 0$

$\Rightarrow$ $m \in (-11, 5)$ ....(ii)

We will combine (i) and (ii), because both must be satisfied

$\Rightarrow$ The common solution is $m \in (-1, 5)$.

Example : 12

Let $y = \sqrt{\frac{2}{x^2 - x + 1} - \frac{1}{x+1} - \frac{(2x+1)}{x^3+1}}$; find all the real values of x for which y takes real values.

Solution

For y to take real values

$$\frac{2}{x^2 - x + 1} - \qquad - \frac{(2x+1)}{x^3+1} \geq 0$$

$$\Rightarrow \qquad \frac{2(x+1) - (x^2 + 1 - x) - (2x+1)}{x^3+1} \geq 0$$

$$\Rightarrow \qquad \frac{-x^2 + x}{(x+1)(x^2 - x + 1)} \geq 0$$

$$\Rightarrow \qquad \frac{x(x-1)}{(x+1)(x^2 - x + 1)} \leq 0$$

As $x^2 - x + 1 > 0$ for all $x \in R$ (because $D < 0$, $a > 0$)

Multiply both sides by $x^2 - x + 1$

$\Rightarrow \quad \frac{x(x-1)}{(x+1)} \le 0$

Critical points are x = 0, x = 1, x = – 1

Expression is negative for

$\Rightarrow \quad x \in (-\infty, -1) \cup [0, 1]$

So real values of x for which y is real are

$x \in (-\infty, -1) \cup [0, 1]$

Example : 13

Find the values of a for which the inequality $(x - 3a)(x - a - 3) < 0$ is satisfied for all x such that $1 \le x \le 3$.

Solution

$(x - 3a)(x - a - 3) < 0$

Case – I :

Let $3a < a + 3 \Rightarrow a < 3/2$ ....(i)

Solution set of given inequality is $x \in (3a, a + 3)$

Now for given inequality to be true for all $x \in [1, 3]$, set [1, 3] should be the subset of (3a, a + 3)

i.e. 1 and 3 lie inside 3a and a + 3 on number line

So we can take, $3a < 1$ and $a + 3 > 3$ ....(ii)

Combining (i) and (ii), we get :

$\Rightarrow \quad a \in (0, 1/3)$

Case – II :

Let $3a > a + 3 \Rightarrow a > 3/2$ ....(iii)

Solution set of given inequality is $x \in (a + 3, 3a)$

As in case–I, [1, 3] should be the subset of (a + 3, 3a)

i.e. $a + 3 < 1$ and $3a > 3$ ....(iv)

Combining (iii) and (iv), we get :

$a \in \{\ \}$ i.e. No solution ....(vi)

Combining both cases, we get : $a \in (0, 1/3)$

Alternate Solution :

Let $f(x) = (x - 3a)(x - a - 3)$

for given equality to be true for all values of $x \in [1, 3]$, 1 and 3 should lie between the roots of f(x) = 0.

$\Rightarrow \quad f(1) < 0$ and $f(3) < 0$ ....[using section 4.1(f)]

Consider $f(1) < 0$:

$\Rightarrow \quad (1 - 3a)(1 - a - 3) < 0$

$\Rightarrow \quad (3a - 1)(a + 2) < 0$

$\Rightarrow \quad a \in (-2, 1/3)$ ....(ii)

Consider $f(3) < 0$:

$\Rightarrow \quad (3 - 3a)(3 - a - 3) < 0$

$\Rightarrow \quad (a - 1)(a) < 0$

$\Rightarrow \quad a \in (0, 1)$ ....(iii)

Combining (ii) and (iii) we get : $a \in (0, 1/3)$

Example : 14

Find all the values of m, for which both the roots of the equation $2x^2 + mx + m^2 - 5 = 0$ are less than 1.

Solution

Let $f(x) = 2x^2 + mx + m^2 - 5$

As both roots of f(x) = 0 are less than 1, we can take $a\,f(1) > 0$, $-b/2a < 1$ and $D \ge 0$

.........[using section 4.1(b)]

Consider $a\,f(1) > 0$:

$\Rightarrow \quad 2[2 + m + m^2 - 5] > 0$

$\Rightarrow \quad m^2 + m - 3 > 0$

$\Rightarrow \quad m \in \left(-\infty, \frac{-1-\sqrt{13}}{2}\right) \cup \left(\frac{-1+\sqrt{13}}{2}, \infty\right)$ ....(i)

Consider – b/2a < 1 :

$\frac{-m}{4} < 1$

$\Rightarrow$ m > – 4 ....(ii)

Consider D ≥ 0 :

$m^2 - 8(m^2 - 5) \geq 0$

$\Rightarrow$ $-7m^2 + 40 \geq 0$

$\Rightarrow$ $7m^2 - 40 \leq 0$

$\Rightarrow$ $m \in \left[-\sqrt{\frac{40}{7}}, \sqrt{\frac{40}{7}}\right]$ ....(iii)

Combining (i), (ii) and (iii) on the number line, we get :

$$m \in \left[-\sqrt{\frac{40}{7}}, \frac{-1-\sqrt{13}}{2}\right) \cup \left(\frac{\sqrt{13}-1}{2}, \sqrt{\frac{40}{7}}\right]$$

Example : 15

Suppose x_1 and x_2 are the roots of the equation $x^2 + 2(k-3)x + 9 = 0$. Find all values of k such that both 6 and 1 lie between x_1 and x_2.

Solution

Let $f(x) = x^2 + 2(k-3)x + 9$

As 1 and 6 lie between x_1 and x_2 , we have

a f (6) < 0, and a f(1) < 0

.... [using section 4.1 (f)]

a f(6) < 0

$\Rightarrow$ 36 + 2 (k – 3) (6) + 9 < 0

$\Rightarrow$ 12k + 9 < 0

$\Rightarrow$ k < –3/4 ....(i)

a f(1) < 0

$\Rightarrow$ 1 + 2 (k – 3) + 9 < 0

$\Rightarrow$ 2k + 4 < 0

$\Rightarrow$ k < – 2

Combining (i), (ii) and (ii) on the number line, we get : $k \in (-\infty, -2)$

Example : 16

If 2, 3 are roots $2x^3 + mx^2 - 13x + n = 0$, find m, n and the third root of the equation.

Solution

Let α be the third root of the equation

Using section 4.2 (d) we can make the following equations,

$\Rightarrow$ $\alpha + 2 + 3 = -m/2$ (sum of roots)

$2\alpha + 3\alpha + 2(3) = -13/2$ (sum of roots taken two at a time)

$2.3 . \alpha = -n/2$ (product of roots)

Hence : $\alpha + 5 = -m/2$ ....(i)

$5\alpha + 6 = -13/2$ ....(ii)

$6\alpha = -n/2$ ....(iii)

Solving (i), (ii), (iii) for α, m and n we get; $\alpha = -5/2$, m = –5, n = 30

Example : 17

Find all the values of p for which the roots of the equation $(p-3)x^2 - 2px + 5p = 0$ are real and positive

Solution

Roots are real and positive if :

D ≥ 0, sum of the roots > 0 and product of the roots > 0

D ≥ 0

$\Rightarrow$ $4p^2 - 20p(p-3) \geq 0$

$\Rightarrow$ $-4p^2 + 15p \geq 0$

$\Rightarrow \quad 4p^2 - 15p \le 0$

$\Rightarrow \quad p \in [0, 15/4]$ ....(i)

Sum of the roots > 0

$\frac{2p}{p-3} > 0 \quad \Rightarrow \quad \frac{p}{p-3} > 0$

$\Rightarrow \quad p\,(p - 3) > 0$

$\Rightarrow \quad p \in (-\infty, 0) \cup (3, \infty)$ ....(ii)

Product of the roots > 0

$\frac{5p}{p-3} > 0$

$\Rightarrow \quad \frac{p}{p-3} > 0$

$\Rightarrow \quad p\,(p - 3) > 0$

$\Rightarrow \quad p \in (-\infty, 0) \cup (3, \infty)$ ....(iii)

Combining (i), (ii) and (iii) on the number line, we get $p \in (3, 15/4]$

Example : 18

If 1, a_1, a_2 ....a_{n-1} are nth roots of unity, then show that $(1 - a_1)(1 - a_2)(1 - a_3)$ .... $(1 - a_{n-1}) = n$.

Solution

The roots of equation $x^n = 1$ are called as the nth roots of unity

Hence 1, a_1, a_2, a_3, ...a_{n-1} are the roots of $x^n - 1 = 0$

$x^n - 1 = (x - 1)(x - a_1)(x - a_2)(x - a_3)$ .... $(x - a_{n-1})$

is an identity in x (i.e., true for all values of x)

$\Rightarrow \quad \frac{x^n - 1}{x - 1} = (x - a_1)(x - a_2)(x - a_3) \ldots\ldots\ldots (x - a_{n-1})$

$\Rightarrow \quad x^{n-1} + x^{n-2} + \ldots\ldots\ldots\ldots + x^0 = (x - a_1)(x - a_2)(x - a_3) \ldots\ldots\ldots (x - a_{n-1})$

[using $x^n - y^n = (x - y)(x^{n-1} y^0 + x^{n-2} y^1 + \ldots\ldots\ldots + x^0 y^{n-1})$]

substituting x = 1 in the above identity, we get;

$n = (1 - a_1)(1 - a_2) \ldots\ldots\ldots\ldots (1 - a_{n-1}) + 0$

$\Rightarrow \quad (1 - a_1)(1 - a_2) \ldots\ldots\ldots (1 - a_{n-1}) = n.$

Example : 19

Solve for x : $|x^2 + 2x - 8| + x - 2 = 0$

Solution

$|x^2 + 2x - 8| + x - 2 = 0$

Case – I

Let $(x - 2)(x + 4) \le 0$

$\Rightarrow \quad x \in [-4, 2]$ ....(i)

the given equation reduces to : $-(x - 2)(x + 4) + x - 2 = 0$

$\Rightarrow \quad x^2 + x - 6 = 0$

$\Rightarrow \quad x = -3, 2$

We accept both the values because they satisfy (i)

Case – II

Let $(x - 2)(x + 4) > 0$

$\Rightarrow \quad x \in (-\infty, -4) \cup (2, \infty)$ ....(ii)

the given equation reduces to : $(x - 2)(x + 4) + x - 2 = 0$

$\Rightarrow \quad (x - 2)(x + 5) = 0$

$\Rightarrow \quad x = -5, 2$

We reject x = 2, because it does not satisfy (ii)

Hence the solution is $x = -5$

Now combining both cases, the values of x satisfying the given equation are $x = -5, -3, 2$.

Example : 20

Solve for x : $x^2 + 2a|x - a| - 3a^2 = 0$ if $a < 0$

Solution

Case – I

Let $x \geq a$ or $x \in [a, \infty)$ and $a < 0$ ....(i)

$\Rightarrow$ the equation is $x^2 + 2a(x - a) - 3a^2 = 0$

$\Rightarrow$ $x^2 + 2ax - 5a^2 = 0$

$\Rightarrow$ $x = -(\sqrt{6} + 1)a, (\sqrt{6} - 1)a$

We reject $(\sqrt{6} - 1)a$ because it does not satisfy (i)

Hence one solution is $-(\sqrt{6} + 1)a$.

Case – II

Let $x < a$ or $x \in (-\infty, a)$ and $a < 0$ ....(ii)

$\Rightarrow$ the equation is $x^2 - 2a(x - a) - 3a^2 = 0$

$\Rightarrow$ $x^2 - 2ax - a^2 = 0$

$\Rightarrow$ $x = (1 + \sqrt{2})a, (1 - \sqrt{2})a$

We reject $x = (1 - \sqrt{2})a$ because it does not satisfy (ii). Hence one solution is $(1 + \sqrt{2})a$

Now combining both cases, we have the final solution as $x = -(\sqrt{6} + 1)a, (1 + \sqrt{2})a$

Example : 21

Solve the following equation for x : $\log_{2x+3}(6x^2 + 23x + 21) + \log_{3x+7}(4x^2 + 12x + 9) = 4$

Solution

$\log_{2x+3}(6x^2 + 23x + 21) + \log_{3x+7}(4x^2 + 12x + 9) = 4$

$\Rightarrow$ $\log_{2x+3}(2x + 3)(3x + 7) + \log_{3x+7}(2x + 3)^2 = 4$

$\Rightarrow$ $1 + \log_{2x+3}(3x + 7) + 2\log_{3x+7}(2x + 3) = 4$ .... [using : log(ab) = loga + logb]

$\Rightarrow$ $\log_{2x+3}(3x + 7) + \dfrac{2}{\log_{2x+3}(3x+7)} = 3$ .... [using $\log_a b = \dfrac{1}{\log_b a}$]

Let $\log_{2x+3}(3x + 7) = t$ ....(i)

$\Rightarrow$ $t + \dfrac{2}{t} = 3$

$\Rightarrow$ $t^2 - 3t + 2 = 0$

$\Rightarrow$ $(t - 1)(t - 2) = 0$

$\Rightarrow$ $t = 1, 2$

Substituting the values of t in (i), we get :

$\log_{2x+3}(3x + 7) = 1$ and $\log_{2x+3}(3x + 7) = 2$

$3x + 7 = 2x + 3$ and $(3x + 7) = (2x + 3)^2$

$\Rightarrow$ $x = -4$ and $4x^2 + 9x + 2 = 0$

$\Rightarrow$ $x = -4$ and $(x + 2)(4x + 1) = 0$

$\Rightarrow$ $x = -4$ and $x = -2, x = -1/4$

As $\log_a x$ is defined for $x > 0$ and $a > 0$ $(a \neq 1)$, the possible values of x should satisfy all of the following inequalities :

$\Rightarrow$ $2x + 3 > 0$ and $3x + 7 > 0$

Also, $(2x + 3) \neq 1$ and $3x + 7 \neq 1$

Out of $x = -4$, $x = -2$ and $x = -1/4$, only $x = -1/4$, only $x = 1/4$ satisfies the above inequalities

So only solution is $x = -1/4$

Example : 22

Solve the following equality for x : $\log_{\left(x+\frac{5}{2}\right)} \left(\frac{x-5}{2x-3}\right)^2 < 0$

Solution

$$\log_{\left(x+\frac{5}{2}\right)} \left(\frac{x-5}{2x-3}\right)^2 < 0$$

If $\log_a b < 0$, then $0 < b < 1$ and $a > 1$ OR $b > 1$ and $0 < a < 1$

Case – I

Let $\left(x+\frac{5}{2}\right) > 1$ and $0 < \left(\frac{x-5}{2x-3}\right)^2 < 1$

Consider $\left(x+\frac{5}{2}\right) > 1$

$\Rightarrow \quad x > -3/2 \quad$ ....(i)

Consider $\left(\frac{x-5}{2x-3}\right)^2 < 1$

$\Rightarrow \quad (x-5)^2 < (2x-3)^2$

$\Rightarrow \quad x^2 + 25 - 10x < 4x^2 + 9 - 12x$

$\Rightarrow \quad 3x^2 - 2x - 16 > 0$

$\Rightarrow \quad (3x-8)(x+2) > 0$

$\Rightarrow \quad x \in (-\infty, -2) \cup (8/3, \infty) \quad$ ....(ii)

Consider $\left(\frac{x-5}{2x-3}\right)^2 > 0$

$\Rightarrow \quad x \in R - \{3/2, 5\} \quad$ ....(iii)

Combining (i), (ii) and (iii), we get :

$x \in (8/3, \infty) - \{5\}$

Case – II

Let : $0 < \left(x+\frac{5}{2}\right) < 1$ and $\left(\frac{x-5}{2x-3}\right)^2 < 1$

Consider : $0 < \left(x+\frac{5}{2}\right) < 1$

$\Rightarrow \quad -\frac{5}{2} < x < -\frac{3}{2} \quad$ ....(iv)

Consider $\left(\frac{x-5}{2x-3}\right)^2 > 1$

$\Rightarrow \quad x \in (-2, 8/3) - \{3/2\} \quad$ ....(v)

Combine (iv) and (v) to get : $x \in \left(-2, -\frac{3}{2}\right)$

Now combining both cases we have the final solution as :

$$x \in \left(-2, -\frac{3}{2}\right) \cup \left(\frac{8}{3}, \infty\right) - \{5\}$$

Example : 23

For what values of the parameter a the equation $x^4 + 2ax^3 + x^2 + 2ax + 1 = 0$ has at least two distinct negative roots.

Solution

The given equation is : $x^4 + 2ax^3 + x^2 + 2ax + 1 = 0$

Divide by x^2 to get : (because, x = 0 does not satisfy the equation)

$$x^2 + 2ax + 1 + \frac{2a}{x} + \frac{1}{x^2} = 0$$

$$\Rightarrow \quad x^2 + \frac{1}{x^2} + 2a\left(x + \frac{1}{x}\right) + 1 = 0$$

Let $\left(x + \frac{1}{x}\right) = t$

$\Rightarrow \quad (t^2 - 2) + 2at + 1 = 0$

$\Rightarrow \quad t^2 + 2at - 1 = 0$

$$\Rightarrow \quad t = \frac{-2a \pm \sqrt{4a^2 + 4}}{2}$$

$$\Rightarrow \quad t = -a \pm \sqrt{a^2 + 1}$$

So we get,

$$x + \frac{1}{x} = -a + \sqrt{a^2 + 1} \text{ and} \qquad \text{............(i)}$$

$$x + \frac{1}{x} = -a - \sqrt{a^2 + 1} \qquad \text{............(ii)}$$

Consider (i)

$$x + \frac{1}{x} = -a + \sqrt{a^2 + 1}$$

$$\Rightarrow \quad x^2 + \left(a - \sqrt{a^2 + 1}\right) x + 1 = 0$$

Sum of the roots = $\sqrt{a^2 + 1} - a$

It can be easily observed that for all $a \in R$ sum of the roots is positive

Product of the roots = $1 > 0$

Product of roots is also positive for all $a \in R$

$\Rightarrow$ As sum of the roots is positive and product of roots is positive, none of the roots is negative

So for given equation to have atleast 2 roots negative both roots of equation (ii) should be negative

Consider (ii)

$$x + \frac{1}{x} = -a - \sqrt{a^2 + 1}$$

$$\Rightarrow \quad x^2 + \left(a + \sqrt{a^2 + 1}\right) x + 1 = 0$$

Sum of roots = $-\left(a + \sqrt{a^2 + 1}\right) < 0$ for all $a \in R$

Product of the roots = $1 > 0$ for all $a \in R$

So for above equation to have both roots negative, D should be positive

i.e. .... [using section 4.1 (g)]

$D > 0$

$$\Rightarrow \quad \left(a + \sqrt{a^2 + 1}\right)^2 - 4 > 0$$

$\Rightarrow \quad \left(a+\sqrt{a^2+1}-2\right)\left(a+\sqrt{a^2+1}+2\right) > 0$

$\Rightarrow \quad \left(a+\sqrt{a^2+1}-2\right) > 0$ .... (As $\left(a+\sqrt{a^2+1}+2\right)$ is positive for all $a \in R$)

$\Rightarrow \quad \sqrt{a^2+1} > 2 - a$ ....(iii)

consider $a < 2 \quad \Rightarrow \quad a^2 + 1 > 4 + a^2 - 4a$

$\Rightarrow \quad 4a > 3 \quad \Rightarrow \quad a > 3/4$ ....(iv)

consider a – 2

$\Rightarrow$ for $a > 2$, RHS < 0 and LHS > 0

$\Rightarrow$ (iii) is true for all $a \geq 2$ ....(v)

Combining (iv) and (v) we get $a > 3/4$

Example : 24

Solve for real x : $x(x^{-} - 1)(x + 2) + 1 = 0$

Solution

$x(x^2 - 1)(x + 2) + 1 = 0$

$\Rightarrow \quad x(x - 1)(x + 1)(x + 2) + 1 = 0$

$\Rightarrow \quad (x^2 + x)(x^2 + x - 2) + 1 = 0$

Let $x^2 + x = y$

$\Rightarrow \quad y(y - 2) + 1 = 0$

$\Rightarrow \quad y^2 - 2y + 1 = 0$

$\Rightarrow \quad (y - 1)^2 = 0$

$\Rightarrow \quad y = 1$

So $\quad x^2 + x - 1 = 0$

$\Rightarrow \quad x^2 + x - 1 = 0$

$\Rightarrow \quad x = \dfrac{-1 \pm \sqrt{5}}{2}$

Example : 25

If each pair of the three equations $x^2 + p_1x + q_1 = 0$, $x^2 + p_2x + q_2 = 0$ and $x^2 + p_3x + q_3 = 0$ have a common roots, then prove that $p_1^2 + p_2^2 + p_3^2 + 4(q_1 + q_2 + q_3) = 2(p_1p_2 + p_2p_3 + p_3p_1)$.

Solution

Since each pair has a common root, the roots of the three equations can be taken as α, β ; β, γ and γ, α respectively.

First equation is : $x^2 + p_1x + q_1 = 0$

$\Rightarrow \quad \alpha + \beta = -p_1$ ....(i)

$\Rightarrow \quad \alpha\beta = q_1$ ....(ii)

Second equation is : $x^2 + p_2x + q_2 = 0$

$\Rightarrow \quad \beta + \gamma = -p_2$ ....(iii)

$\Rightarrow \quad \beta\gamma = q_2$ ....(iv)

Third equation is : $x^2 + p_3x + q_3 = 0$

$\Rightarrow \quad \alpha + \gamma - p_3$ ....(v)

$\Rightarrow \quad \alpha\gamma = q_3$ ....(vi)

On adding (i), (iii) and (v), we get :

$2(\alpha + \beta + \gamma) = -(p_1 + p_2 + p_3)$ ....(vii)

To prove that :

$p_1^2 + p_2^2 + p_3^2 + 4(p_1 + p_2 + p_3) = 2(p_1p_2 + p_2p_3 + p_3p_1)$

To prove that :

$p_1^2 + p_2^2 + p_3^2 = 2(p_1p_2 + p_2p_3 + p_3p_1) - 4(p_1 + p_2 + p_3)$

Add $2(p_1p_2 + p_2p_3 + p_3p_1)$ to both sides, we get :

$(p_1 + p_2 + p_3)^2 = 4(p_1p_2 + p_2p_3 + p_3p_1 - q_1 - q_2 - q_3)$

Consider RHS

$RHS = 4(p_1p_2 + p_2p_3 + p_3p_1 - q_1 - q_2 - q_3)$

Using (ii), (iv) and (vi), we get :

$= 4[(\alpha + \beta)(\beta + \gamma) + (\beta + \gamma)(\alpha + \gamma) + (\alpha + \gamma)(\alpha + \beta) - \alpha\beta - \beta\gamma - \gamma\alpha]$

$= 4(\alpha^2 + \beta^2 + \gamma^2 + 2\alpha\beta + 2\alpha\gamma + 2\beta\gamma)$

$= 4(\alpha + \beta + \gamma)^2$ .... [using (7)]

$= (p_1 + p_2 + p_3)^2 = \text{LHS}$

Example : 26

Solve for real x : $x^2 + \dfrac{x^2}{(x+1)} = 3$

Solution

Use : $a^2 + b^2 = (a - b)^2 + 2ab$ to get

$$\left(x - \frac{x}{x+1}\right)^2 + \frac{2x^2}{(x+1)} - 3 = 0.$$

$$\Rightarrow \quad \left(\frac{x^2 + x - x}{x+1}\right)^2 + \frac{2x^2}{(x+1)} - 3 = 0$$

$$\Rightarrow \quad \left(\frac{x^2}{x+1}\right)^2 + \frac{2x^2}{(x+1)} - 3 = 0$$

Let $\dfrac{x^2}{(x+1)} = y$

$\Rightarrow \quad y^2 + 2y - 3 = 0$

$\Rightarrow \quad y = 1, -3$

$\Rightarrow \quad \dfrac{x^2}{(x+1)} = 1$ and $\dfrac{x^2}{(x+1)} = -3$

$\Rightarrow \quad x^2 + x - 1 = 0$ and $x^2 + 3x + 3 = 0$

$\Rightarrow \quad x = \dfrac{1 \pm \sqrt{5}}{2}$ and No real roots (D < 0)

So possible values of x are $\dfrac{1 \pm \sqrt{5}}{2}$

Example : 27

Solve for x : $2^{|x+1|} - 2^x = |2^x - 1| + 1$

Solution

Find critical points

$x + 1$ and $2^x - 1 = 0$

$\Rightarrow \quad x = -1$ and $x = 0$

so critical points are $x = 0$ and $x = -1$

Consider following cases :

$x \le -1$ ....(i)

$2^{-(x+1)} - 2^x = -(2^x - 1) + 1$

$2^{-x-1} - 2^x = -2^x + 2$

$\Rightarrow \quad 2^{-x-1} = 2$

$\Rightarrow \quad -x - 1 = 1$

$\Rightarrow \quad x = -2$

As $x = -2$ satisfies (i), one solution is $x = -2$

$-1 < x \le 0$ ....(ii)

$2^{x+1} - 2^x = -(2^x - 1) + 1$

$\Rightarrow \quad 2^{x+1} = 2$

$\Rightarrow \quad x + 1 = 1$

$\Rightarrow \quad x = 0$

As $x = 0$ satisfies (ii), second solution is $x = 0$

$x > 0$ ....(iii)

$2^{x+1} - 2^x = (2^x - 1) + 1$

$\Rightarrow$ $2^{x+1} = 2^{x+1}$

$\Rightarrow$ identity in x, i.e. true for all $x \in R$

On combining $x \in R$ with (iii), we get :

$x > 0$

Now combining all cases, we have the final solution as :

$x \geq 0$ and $x = -2$