Example : 1

A straight line drawn through point A (2, 1) making an angle π/4 with the +X-axis intersects another line x + 2y + 1 = 0 in point B. Find the length AB.

Solution

Let AB = r

from parametric form, the point B can be taken as :

B = (x_A + r cos θ, y_A + r sin θ)

B = (2 + r cos π/4,1 + r sin π /4)

B = (2 + r/$\sqrt{2}$, 1 + r/$\sqrt{2}$)

As B lies on x + 2y + 1 = 0, we have $\left(2+\frac{r}{\sqrt{2}}\right) + 2\left(1+\frac{r}{\sqrt{2}}\right) = -1$

⇒ $r = -\frac{5\sqrt{2}}{3}$

r is negative because the point B lies below the point A.

⇒ $AB = \frac{5\sqrt{2}}{3}$

Alternative Method :

Find the equation of AB from point-slope form and then solve with x + 2y + 1 = 0 simultaneously to get coordinates of AB. Then use distance formula to find AB.

Example : 2

If two opposite vertices of a square are (1, 2) and (5, 8), find the coordinates of its other vertices.

Solution

Let ABCD be the square and A ≡ (1, 2) and C ≡ (5, 8)

Let P be the intersection of diagonals

⇒ P ≡ [(1 + 5)/2, (2 + 8)/2]

⇒ P ≡ (3, 5)

To find B and D, we will apply parametric form for the line BD with P as the given point

$PB = PD = \frac{1}{2}\ AC = \frac{1}{2}\sqrt{(8-2)^2+(5-1)^2}$

⇒ $PB = PD = \sqrt{13}$

$\text{Slope (AC)} = \frac{8-2}{5-1} = \frac{3}{2}$

⇒ $\text{slope (BD)} = -\frac{2}{3} = \tan\theta$ ⇒ tan θ is obtuse

Where θ is the angle between BD and +ve X-axis

⇒ $\cos\theta = -\frac{3}{\sqrt{13}}$ and $\sin\theta = \frac{2}{\sqrt{13}}$

using parametric form on BD with P ≡ (x_1, y_1) ≡ (3, 5)

Coordinates of D :

$r = +\sqrt{13}$ because D is above P.

⇒ D ≡ (x_1 + r cos θ, y_1 + r sin θ)

⇒ $D \equiv \left[3+\sqrt{13}\left(-\frac{3}{\sqrt{13}}\right), 5+\sqrt{13}\frac{2}{\sqrt{13}}\right]$

⇒ D ≡ (0, 7)

Coordinates of B :

$r = -\sqrt{13}$ because B is below P.

$\Rightarrow$ $B \equiv (x_1 + r\cos\theta, y_1 + r\sin\theta)$

$$\Rightarrow \quad B \equiv \left[3 - \sqrt{13}\left(-\frac{3}{\sqrt{13}}\right) 5 - \sqrt{13}\,\frac{2}{\sqrt{13}}\right]$$

$\Rightarrow$ $B \equiv (6, 3)$

Example : 3

Two opposite vertices of a square are (1, 2) and (5, 8). Find the equations of its side.

Solution

Let ABCD be the square

m = slope of AC = (8 – 2) / (5 – 1) = 3/2

lines AB and AD make an angle $\alpha = 45^o$ with AC

$$m_1 = \text{slope (AD)} = \frac{m + \tan\alpha}{1 - m\,.\,\tan\alpha} = \frac{3/2 + \tan 45^o}{1 - 3/2\,.\,\tan 45^o} = -5$$

$$m_2 = \text{slope (AB)} = \frac{m - \tan\alpha}{1 + m\,.\,\tan\alpha} = \frac{3/2 - \tan 45^o}{1 + 3/2 \tan 45^o} = \frac{1}{5}$$

We also have AB || DC and AB || DC.

$\Rightarrow$ slope (DC) = 1/5 and slope (BC) = – 5

Now use $y - y_1 = \text{slope}\,(x - x_1)$ on each side

Equation of AB :

$y - 2 = 1/5\,(x - 1) \quad \Rightarrow \quad x - 5y + 9 = 0$

Equation of AD :

$y - 2 = -5\,(x - 1) \quad \Rightarrow \quad 5x + y - 7 = 0$

Equation of BC :

$y - 8 = -5\,(x - 5) \quad \Rightarrow \quad 5x + y - 33 = 0$

Equation of CD :

$y - 8 = 1/5\,(x - 5) \quad \Rightarrow \quad x - 5y + 35 = 0$

Alternative Method :

Find the coordinates of B and D on the pattern of illustration and then use two-point form of equation of line for each side.

Example : 4

The equation of the base of an equilateral triangle is x + y = 2 and its vertex is (2, – 1). Find the length and equations of its sides.

Solution

Let $A \equiv (2, -1)$ and B, C be the other vertices of the equilateral triangle. Length of the perpendicular from A to BC (x + y – 2 = 0)

$$\Rightarrow \quad p = \frac{|2 + (-1) - 2|}{\sqrt{1^2 + 1^2}} = \frac{1}{\sqrt{2}}$$

$$\text{Side} = \frac{p}{\sin 60^o} = \frac{1}{\sqrt{2}} \times \frac{2}{\sqrt{3}} = \sqrt{\frac{2}{3}}$$

Now AB and AC make equal angles $\alpha = 60^o$ with line BC whose slope is m = – 1

$$m_1 = \text{slope (AC)} = \frac{m + \tan\alpha}{1 - m, \tan\alpha} = \frac{(-1) + \tan 60^o}{1 - (-1)\,.\,\tan 60^o} = 2 - \sqrt{3}$$

$$m_2 = \text{slope of (AB)} = \frac{m - \tan\alpha}{1 - m\,.\,\tan\alpha} = \frac{-1 - \tan 60^o}{1 + (-1)\,.\,\tan 60^o} = 2 + \sqrt{3}$$

Equation of AC :

$$y = (-1) = (2 - \sqrt{3})\,(x - 2)$$

$\Rightarrow \quad (2 - \sqrt{3})\, x - y - 5 + 2\sqrt{3} = 0$

Equation of AB :

$$y - (-1) = (2 + \sqrt{3})\,(x - 2)$$

$\Rightarrow \quad (2 + \sqrt{3})\, x - y - 5 - 2\sqrt{3} = 0$

Example : 5

Find the equations of straight lines passing through (–2, –7) and having an intercept of length 3 between the straight lines : 4x + 3y = 12, 4x + 3y = 3.

Solution

Let the required line cut the given parallel lines in points A and B.

$\Rightarrow \quad AB = 3$

Let AC be the perpendicular distance between the given lines

$$\Rightarrow \quad AC = \frac{|12 - 3|}{\sqrt{4^2 + 3^2}} = \frac{9}{5}$$

$$\Rightarrow \quad \sin\theta = \frac{AC}{AB} = \frac{9/5}{3} = \frac{3}{5}$$

hence the required line(s) cut the given parallel lines at an angle θ where :

$\sin\theta = 3/5 \quad \Rightarrow \quad \tan\theta = 3/4$

Let m_1 and m_2 be the slopes of required lines.

Slopes of the given parallel lines = m = –4/3

$$m_1 = \frac{m + \tan\theta}{1 - m\tan\theta} = \frac{-4/3 - 3/4}{1 + 4/3\,.\,3/4} = -\frac{7}{24}$$

$$m_2 = \frac{m - \tan\theta}{1 + m\,.\,\tan\theta} = \frac{-4/3 - 3/4}{1 - 4/3\,.\,3/4} = \text{undefined.}$$

Hence one line is parallel to Y-axis and passes through (–2, –7)

$\Rightarrow$ its equation is : y + 7 = –7/24 (x + 2)

$\Rightarrow$ 7x + 24y + 182 = 0

Example : 6

Two straight lines 3x + 4y = 5 and 4x – 3y = 15 intersect at point A. Points B and C are chosen on these two lines, such that AB = AC. Determine the possible equations of the line BC passing through the point (1, 2).

Solution

Through the point (1, 2) two lines L_1 and L_2 can be drawn and

hence two equations are possible for line BC.

Let m be the slope of BC

AB = BC $\quad \Rightarrow \quad$ ΔABC is isosceles and hence acute angle between BC and AB is equal to the acute angle between BC and AC.

Acute angle between AB(3x 4y = 5) and BC is α :

$$\tan\alpha = \left|\frac{m - (-3/4)}{1 + m(-3/4)}\right|$$

Acute angle between AC (4x – 3y = 15) and BC is α :

$$\tan\alpha = \left|\frac{m - (4/3)}{1 + m(-3/4)}\right|$$

$$\Rightarrow \quad \left|\frac{m - (-3/4)}{1 + m(-3/4)}\right| = \left|\frac{m - (4/3)}{1 + m(4/3)}\right|$$

$$\Rightarrow \quad \frac{4m + 3}{4 - 3m} = \pm\frac{3m - 4}{3 + 4m} = \pm\frac{m - (4/3)}{1 + m(4/3)}$$

Taking + sign
(4m + 3) (3 + 4m) = (3m – 4) (4 – 3m)
$16m^2 + 24m + 9 = -9m^2 + 24m - 16$
$25m^2 = -25$ which is impossible
Taking + sign
(4m + 3) (3 + 4m) = –(3m – 4) (4 – 3m)
$16m^2 + 24m + 9 = 9m^2 - 24m + 16$
⇒ $7m^2 + 48m - 7 = 0$
⇒ (m + 7) (7m + 1) = 0
⇒ m = – 7 or m = 1/7
Equation of BC are :
y – 2 = –7 (x – 1) and y – 2 = 1/7 (x – 1)
⇒ 7x + y – 9 = 0 and x – 7y + 13 = 0
Method 2 ;
As line BC makes equal angles with AB and AC, it must be parallel to one of the angle bisectors of AB and AC. By finding the equations of bisectors, we get the slope of BC.
Angle bisectors of AB and AC are :

$$\frac{3x-4y-5}{\sqrt{9+16}} = \pm \frac{4x-3y-15}{\sqrt{16+9}}$$

⇒ x – 7y – 10 = 0 and 7x + y – 20 = 0
⇒ slopes are 1/7 and – 7
⇒ slopes of BC are m = 1/7 and m = –7
⇒ Equations are BC are
y – 2 = 1/7 (x – 1) and y – 2 = –7 (x – 1)
⇒ 7x + y – 9 = 0 and x – 7y + 13 = 0

Example : 7
Lines $L_1 \equiv ax + by + c = 0$ and $L_2 \equiv \ell x + my + n = 0$ intersect at point P and make an angle θ with each other. Find the equation of the line L different from L_2 which passes through P makes the same angle with L_1 .
Solution
As L passes through the intersection of L_1 and L_2 , let its equation be :
$(ax + by + c) + k (\ell x + my + n) = 0$ ....(i)
where k is a parameter
As L_1 is the angle bisector of L and L_2 , any arbitrary point $A(x_1, y_1)$ on L_1 is equidistant from L and L_2 .

$$\Rightarrow \quad \frac{\ell x_1 + my_1 + n|}{\sqrt{\ell^2+m^2}} = \frac{|ax_1+by_1+c+k(\ell x_1+my_1+n)}{\sqrt{(a+k\ell)^2+(b+km)^2}}$$

But A lies on L_1 . hence it must satisfy the equation of L_1
⇒ $ax_1 + by_1 + c = 0$

$$\Rightarrow \quad \frac{|\ell x_1+my_1+n|}{\sqrt{\ell^2+m^2}} = \frac{|0+(\ell x_1+my_1+n)}{\sqrt{(a+k\ell)^2+(b+km)^2}}$$

⇒ $k^2 (\ell^2 + m^2) = (a + k\ell)^2 + (b + km)^2$

$$\Rightarrow \quad k = -\frac{a^2+b^2}{2a\ell+2bm}$$

$$\Rightarrow \quad (ax + by + c) - \left(\frac{a^2+b^2}{2a\ell+2bm}\right)(\ell x + my + n) = 0 \text{ is the equation of L.}$$

⇒ $(2a\ell + 2bm) (ax + by + c) - (a^2 + b^2) (\ell x + my + n) = 0$
Alternative Method :
Let S be the slope of line L.

$\Rightarrow \quad \tan\theta = \left|\dfrac{S-(-a/b)}{1+S(-a/b)}\right| = \left|\dfrac{(-\ell/m)-(-a/b)}{1+\dfrac{\ell a}{mb}}\right|$

($\because$ by taking +ve sign, we will get S = –ℓ/m which is not the slope of L)

We also have $S = -\left(\dfrac{a+k\ell}{b+km}\right)$ [equation (i)]

Substituting for S, we can value of k.

Example : 8

Find all points on x + y = 4 that lie at a unit distance from the line 4x + 3y – 10 = 0

Solution

Let P (t, 4 – t) be an arbitrary point on the line x + y = 4
distance of P from 4x + 3y – 10 = 0 is unity

$\Rightarrow \quad \dfrac{|4t+3(4-t)-10|}{\sqrt{16+9}} = 1$

$\Rightarrow$ |t + 2| = 5
$\Rightarrow$ t = – 2 ± 5 = –7, 3
$\Rightarrow$ points are (–7, 11) and (3, 1)
Draw the diagram yourself

Example : 9

One side of a rectangle lies on the line 4x + 7y + 5 = 0. Two of its vertices are (–3, 1) and (1, 1). Find the equations of other three sides.

Solution

One side is 4x + 7y + 5 = 0

$\Rightarrow$ slope of the four sides of rectangle are : $-\dfrac{4}{7}, \dfrac{7}{4}, -\dfrac{4}{7}, \dfrac{7}{4}$

Slope of Line joining (–3, 1) and (1, 1) = $\dfrac{1-1}{1+3} = 0$

Hence A(–3, 1) and C(1, 1) are opposite vertices. Let ABCD be the rectangle with AB lying along 4x + 7y + 5 = 0 (check that A lies on this line)

Equation of AD :
y – 1 = 7/4 (x + 3)
$\Rightarrow$ 7x – 4y + 25 = 0

Equation of CB :
y – 1 = 7/4 (x – 1)
$\Rightarrow$ 7x – 4y – 3 = 0

Equation of CD :
y – 1 = – 4/7 (x – 1)
$\Rightarrow$ 4x + 7y – 11 = 0

Example : 10

Find the coordinates of incentre of the triangle formed by 3x – 4y = 17; y = 4 and 12x + 5y = 12.

Solution

Let A, B and C be the vertices of the triangle Let us first find the equation of interior angle bisectors of the triangle ABC. The coordinates of vertices can be calculate as :
A ≡ (19/9, –8/3), B ≡ (11, 4) and C ≡ (–2/3, 4)
Interior Bisector of angle A :
bisectors of AB and AC are :

$$\frac{3x-4y-17}{5} = \pm\frac{12x+5y-12}{13}$$

21x + 77y + 161 = 0 and 99x – 27y – 281 = 0

$\Rightarrow$ $3x + 11y + 23 = 0$ and $99x - 27y - 281 = 0$

B and C must lie on opposite sides of the interior bisector

Consider $3x + 11y + 23 = 0$

for $B \equiv (11, 4)$: LHS = 3(11) + 11(4) + 23 = 100

for $C \equiv (-2/3, 4)$: LHS = – 2 + 44 + 23 = 65

Both have same sign and hence B, C are one same side.

$\Rightarrow$ this is exterior bisector.

Hence the interior bisector of angle A is :

$99x - 27y - 281 = 0$ ....(i)

Interior bisector of angle B :

following the same procedure, we get the equation of interior bisector of B as :

$3x + 9y + 3 = 0$ ....(ii)

Solving (i) and (ii) simultaneously, we get the coordinates of incentre :

$$I = \left(\frac{29}{9}, \frac{38}{27}\right)$$

Example : 11

The ends AB of a straight line segment of constant length C slide upon the fixed rectangular axes OX and OY respectively. If The rectangle OAPB be completed, then show that the locus of the foot of perpendicular drawn from P to AB is $x^{2/3} + y^{2/3} = C^{2/3}$.

Solution

Let $A \equiv (a, 0)$ and $B \equiv (0, b)$

$\Rightarrow$ $P \equiv (a, b)$

$PQ \perp AB$

We have to find the locus of the point Q.

Let $Q \equiv (x_1, y_1)$

$AB = C \Rightarrow a^2 + b^2 = c^2$ ....(i)

$PQ \perp AB \Rightarrow$ slope (PQ) × slope (AB) = – 1

$$\Rightarrow \left(\frac{b - y_1}{a - x_1}\right) \times \left(\frac{0 - b}{a - 0}\right) = -1$$

$\Rightarrow$ $ax_1 - by_1 = a^2 - b^2$ ....(ii)

Q lies on AB whose equation is $\frac{x}{a} + \frac{y}{b} = 1$

$$\Rightarrow \frac{x_1}{a} + \frac{y_1}{b} = 1$$

$\Rightarrow$ $bx_1 + ay_1 = ab$ ....(iii)

In the problem, C is a fixed quantity while a, b are changing, we will eliminate a, b from (i), (ii) and (iii) to get the locus. By solving (ii) and (iii), we get :

$$x_1 = \frac{a^3}{a^2 + b^2} \quad \text{and} \quad y_1 = \frac{b^3}{a^2 + b^2}$$

consider $$x_1^{2/3} + y_1^{2/3} = \frac{b^2 + a^2}{(a^2 + b^2)^{2/3}} = (a^2 + b^2)^{1/3} = (C)^{2/3}$$

$\Rightarrow$ $x_1^{2/3} + y_1^{2/3} = C^{2/3}$

$\Rightarrow$ $x^{2/3} + y^{2/3} = C^{2/3}$ is the equation of required locus.

Alternative method :

Let angle OAB = θ

$\Rightarrow$ $OA = C \cos\theta$ and $OB = C \sin\theta = AP$

From ΔAPQ :

$AQ = (C \sin\theta) \sin\theta = C\sin^2\theta$

Draw $QM \perp OA$

From ΔAQM :

$AM = AQ \cos\theta = C \sin^2\theta \cos\theta$

$QM = AQ \sin\theta = C \sin^3\theta$

$QM = y_1 = C \sin^3\theta$

and $OM = x_1 = OA - AM = C \cos\theta - C \sin^2\theta \cos\theta$

$\Rightarrow \quad x_1 = C \cos\theta (1 - \sin^2\theta) = C \cos^3\theta$

$\Rightarrow \quad x_1 = C \cos^3\theta$ and $y_1 = C \sin^3\theta$. We will eliminate θ

Substituting for $\cos\theta$, $\sin\theta$ in $\sin^2\theta + \cos^2\theta = 1$, we get :

$$\left(\frac{x_1}{C}\right)^{2/3} + \left(\frac{y_1}{C}\right)^{2/3} = 1$$

$\Rightarrow \quad x_1^{2/3} + y_1^{2/3} = C^{2/3}$

$\Rightarrow \quad x^{2/3} + y^{2/3} = C^{2/3}$ is the locus of Q.

Example : 12

A variable line is draw through O to cut two fixed straight lines L_1 and L_2 in R and S. A point P is chosen on the variable line such that : $\frac{m+n}{OP} = \frac{m}{OR} + \frac{n}{OS}$. Show that the locus of P is a straight line passing through intersection of L_1 and L_2

Solution

Let the fixed point O be at origin.

Let $L_1 \equiv ax + by + c = 0$, $L_2 \equiv Lx + My + N = 0$ and $P \equiv (x_1, y_1)$

As lines L_1 and L_2 are fixed, (a, b, c, L, M, N) are fixed quantities.

Parametric form is likely to be used because distance of P, R and S from a fixed point are involved.

Let θ be the angle made by the variable line ORS with + ve X-axis. Note that θ is a changing quantity and we will have to eliminate it later

Let $OR = r_1$; $OS = r_2$ and $OP = r$

Note that r_1, r_2 r are also changing quantities.

Using parametric form, we have :

$R \equiv (r_1 \cos\theta, r_1 \sin\theta)$, $S \equiv (r_2 \cos\theta, r_2 \sin\theta)$

$P \equiv (r \cos\theta, r \sin\theta) \equiv (x_1, y_1)$

As R lies on L_1 , $ar_1 \cos\theta + br_1 \sin\theta + c = 0$

$\Rightarrow \quad r_1 = \dfrac{-c}{a\cos\theta + b\sin\theta}$

As S lies on L_2 , $Lr_2 \cos\theta + Mr_2 \sin\theta + N = 0$

$\Rightarrow \quad r_2 = \dfrac{-N}{L\cos\theta + M\sin\theta}$

Substituting in $\dfrac{m+n}{OP} = \dfrac{m}{OR} + \dfrac{n}{OS}$

$$\Rightarrow \quad \frac{m+n}{r} = \frac{m}{r_1} + \frac{n}{r_2}$$

$$\Rightarrow \quad \frac{(m+n)}{r} = -\frac{m(a\cos\theta + b\sin\theta)}{c} - \frac{n(L\cos\theta + M\sin\theta)}{N}$$

Put $\cos\theta = \dfrac{x_1}{r}$ and $\sin\theta = \dfrac{y_1}{r}$ to eliminate θ

$$\Rightarrow \quad \frac{m+n}{r} = -\frac{m}{c}\left[\frac{ax_1}{r} + \frac{by_1}{r}\right] - \frac{n}{N}\left[\frac{Lx_1}{r} + \frac{My_1}{r}\right]$$

$$\Rightarrow \quad (m + n) = -\frac{m}{c}(ax_1 + by_1) - \frac{n}{N}(Lx_1 + My_1)$$

$$\Rightarrow \quad (ax_1 + by_1 + c) + \frac{nc}{mN}(Lx_1 + My_1 + N) = 0$$

The above equation is the locus of P which represents a straight line passing through the intersection of L_1 and L_2

Example : 13

Let (h, k) be a fixed point, where h > 0, k > 0. A straight line passing through this point cuts the positive direction of the coordinates axes at the points P and Q. Find the minimum area of the triangle OPQ, O being the origin.

Solution

Equation of any line passing through the fixed point (h, k) and having slope m can be taken as :

$y - k = m(x - h)$ ....(i)

Put y = 0 in (i) to get OP i.e. $X_{intercept} = OP = h - \frac{k}{m}$

Put x = 0 in (i) to get OQ i.e. $Y_{intercept} = OQ = k - mh$

Area of triangle OPQ = $A(m)\ \frac{1}{2}\left(h - \frac{k}{m}\right)(k - mh) = \frac{1}{2}\left(2hk - mh^2 - \frac{k^2}{m}\right)$

$\Rightarrow \quad A(m) = \frac{1}{2}\left(2hk - mh^2 - \frac{k^2}{m}\right)$ ....(ii)

To minimise A(m), Put A′ (m) = 0

$\Rightarrow \quad A'(m) = \frac{1}{2}\left(-h^2 + \frac{k^2}{m^2}\right) = 0 \quad \Rightarrow \quad m = \pm\frac{k}{h}$

$A''(m) = -\frac{k^2}{m^2} \quad \Rightarrow \quad A''\left(\frac{-k}{h}\right) = \frac{h^3}{k} > 0$

$\Rightarrow$ for m = –k/h, A(m) is minimum.

Put m = –k/h is (ii) to get minimum area.

$\Rightarrow$ Minimum Area of ΔOPQ = $\frac{1}{2}$ [2hk + kh + hk[= 2hk]

Example : 14

A rectangle PQRS has its side PQ parallel to the line y = mx and vertices P, Q and S lie on the lines y = a, x = b and x = – b, respectively. Find the locus of the vertex R.

Solution

Let coordinates of P be (t, a) and R be (x_1, y_1)

Slope of PQ = m (given)

Slope of PS = – 1/(slope of PQ) = –1/m

Equation of Pθ ≡ y – a = m (x – t) ....(i)

As Q lies on x = b line, put x = b in (i) to get Q.

$\Rightarrow$ Q ≡ [b, a + m (b – t)]

Equation of PS ≡ y – a = – 1/m (x – t) ....(ii)

As S lies on x = – b line, put x = –b in (ii) to get S.

$\Rightarrow$ S ≡ [–b, a + 1/m (b + t)]

Slope of RS = $\frac{y_1 - a\frac{-1}{m}(b+t)}{x_1 + b} = m$ ....(iii)

$\Rightarrow$ b + t = m $(y_1 - a) - m^2 (x + b)$

Slope of RQ = $\frac{y_1 - a - m(b-t)}{x_1 - b} = -\frac{1}{m}$

$\Rightarrow \quad \frac{m(y_1 - a) + (x_1 - b)}{m^2} = b - t$

Add (iii) and (iv) to eliminate t

$\Rightarrow \quad 2b = m\,(y_1 – a) – m^2\,(x + b) + \dfrac{m(y_1 - a) + (x_1 - b)}{m^2}$

$\Rightarrow \quad$ Locus is : $my + (1 – m^2)\,x – am – b\,(1 + m^2) = 0$

Example : 15

Let ABC be a triangle with AB = AC. If D is the midpoint of BC, E the foot of the perpendicular drawn from D to AC and F the midpoint of DE, prove the AF is perpendicular to BE.

Solution

Let vertex A of the triangle be at origin and AC as x-axis. Let the coordinates of C and B be (4a, 0) and (4b, 4c) respectively.

Then the coordinates of points D, E and F will be (2a + 2b, 2c), (2a + 2b, 0) and (2a + 2b, c) respectively.

Since AB = AC, we will have $(4c)^2 + (4b)^2 = (4a^2)$

$\Rightarrow \quad b^2 + c^2 = a^2 \quad$ ....(i)

Now, Slope of BE = $\dfrac{0 - 4c}{(2a + 2b) - 4b} = \dfrac{2c}{b - a}$

Slope of AF = $\dfrac{c - 0}{(2a + 2b) - 0} = \dfrac{c}{2(b + a)}$

Slope of BE × AF = $\dfrac{c^2}{b^2 - a^2} = – 1$ [using (i)]

Hence AF ⊥ BE

Example : 16

A line through A (–5, –4) meets the lines x + 3y + 2 = 0, 2x + y + 4 = 0 and x – y – 5 = 0 at the points B, C and D respectively. If $(15/AB)^2 + (10/AC)^2 = (6/AD)^2$, find the equation of the line.

Solution

The parametric form of the line passing through A(–5, –4) is

$x = – 5 + r\cos\theta$

$y = –4 + r\sin\theta \quad$ ....(i)

where r is the distance of any other point P(x, y) on this line from A.

Equation (i) meets the line x + 3y + 2 = 0 at B.

Let AB = r_1

$\Rightarrow$ The coordinates of B are $(–5 + r_1\cos\theta, – 4 + r_1\sin\theta)$

Since B lies on x + 3y + 2 = 0, we get

$(–5 + r_1\cos\theta) + 3\,(–4 + r_1\sin\theta) + 2 = 0$

$\Rightarrow \quad r_1 = \dfrac{15}{\cos\theta + 3\sin\theta} \quad$ ....(ii)

Equation (i) meets the line 2x + y + 4 = 0 at C.

Let Ac = r_2

$\Rightarrow$ The coordinates of C are $(–5 + r_2\cos\theta, –4 + r_2\sin\theta)$

Since C lies on 2x + y + 4 = 0, we get

$2(–5 + r_2\cos\theta) + (–4 + r_2\sin\theta) + 4 = 0$

$\Rightarrow \quad r_2 = \dfrac{10}{2\cos\theta + \sin\theta} \quad$ ....(iii)

Similarly, $r_3 = \dfrac{6}{\cos\theta - \sin\theta}$ where r_3 = AD ....(iv)

It is given that : $\left(\dfrac{15}{AB}\right)^2 + \left(\dfrac{10}{AC}\right)^2 = \left(\dfrac{6}{AD}\right)^2$

$\Rightarrow \quad \left(\dfrac{15}{r_1}\right)^2 + \left(\dfrac{10}{r_2}\right)^2 = \left(\dfrac{6}{r_3}\right)^2$

Substituting r_1, r_2 and r_3 from equation (ii), (iii) and (iv), we get

$$(\cos\theta + 3\sin\theta)^2 + (2\cos\theta + \sin\theta)^2 = (\cos\theta - \sin\theta)^2$$

$\Rightarrow$ $(\cos^2\theta + 9\sin^2\theta + 6\cos\theta\sin\theta) + (4\cos^2\theta + \sin^2\theta + 4\cos\theta\sin\theta)$

$= \cos^2\theta + \sin^2\theta - 2\cos\theta\sin\theta$

$\Rightarrow$ $4\cos^2\theta + 9\sin^2\theta + 12\cos\theta\sin\theta = 0$

$\Rightarrow$ $(2\cos\theta + 3\sin\theta)^2 = 0$

$\Rightarrow$ $2\cos\theta + 3\sin\theta = 0$ $\Rightarrow$ $\tan\theta = -2/3$

$\Rightarrow$ slope of the line $= -2/3$

Hence equation of required line is : $y + 5 = -2/3\ (x + 4)$

$\Rightarrow$ $3y + 2x + 23 = 0$

Example : 17

Using the methods of co-ordinates geometry, show that $\frac{BP}{PC}\cdot\frac{CQ}{QA}\cdot\frac{AR}{RB} = -1$, where P, Q, R the points of intersection of a line L with the sides BC, CA, AB of a triangle ABC respectively.

Solution

Let A (x_1, y_1), B (x_2, y_2) and C(x_3, y_3) be the vertices of the ΔABC.

Let the equation of the line L be $ax + by + c = 0$

Let L divide BC at P in the ratio m : 1 i.e. $\frac{BP}{PC} = \frac{m}{1}$

Using section formula, the coordinates of P are $\left(\frac{x_1 + mx_3}{1+m}, \frac{y_2 + my_3}{1+m}\right)$

As P lies on the line L

$$a\left(\frac{x_2 + mx_3}{1+m}\right) + b\left(\frac{y_2 + my_3}{1+m}\right) + c = 0$$

$\Rightarrow$ $m(ax_3 + by_3 + c) + (ax_2 + by_2 + c) = 0$

$\Rightarrow$ $\frac{m}{1} = -\left(\frac{ax_2 + by_2 + c}{ax_3 + by_3 + c}\right)$

$\Rightarrow$ $\frac{BP}{PC} = -\left(\frac{ax_2 + by_2 + c}{ax_3 + by_3 + c}\right)$ ....(i)

Similarly $\frac{CQ}{QA} = \frac{ax_3 + by_3 + c}{ax_1 + by_1 + c}$ ....(ii)

and $\frac{AR}{RB} = -\frac{ax_1 + by_1 + c}{ax_2 + by_2 + c}$ ....(iii)

Multiple (i), (ii) and (iii) to get : $\frac{BP}{PC}, \frac{CQ}{QA}, \frac{AR}{RB} = -1$

Example : 18

The vertices of a triangle are $A(x_1, x_1 \tan\theta_1)$, $B(x_2, x_2\tan\theta_2)$, and $C(x_3, x_3\tan\theta_3)$. If the circumcentre of ΔABC coincides with the origin and $H(x', y')$ is the orthocentre, show that : $\frac{y'}{x'} = \frac{\sin\theta_1 + \sin\theta_2 + \sin\theta_3}{\cos\theta_1 + \cos\theta_2 + \cos\theta_3}$

Solution

Let circumcentre of the triangle ABC = r

Since origin is the circumcentre of ΔABC, OA = OB = OC = r

Using Distance Formula,

$$x_1^2 + x_1^2\tan^2\theta_1 = x_2^2 + x_2^2\tan^2\theta_2 = x_3^2 + x_3^2\tan^2\theta_3$$

$\Rightarrow$ $x_1\sec\theta_1 = x_2\cos\theta_2 = x_3 = \sec\theta_3 = r$

$\Rightarrow \quad x_1 = r\cos\theta_1\ ,\ x_2 = \sec\theta_2\ ,\ x_3 = r\cos\theta_3$

Therefore, the coordinates of the vertices of the triangle are :

$A \equiv (r\cos\theta_1, r\sin\theta_1)$

$B \equiv (r\cos\theta_2, r\sin\theta_2)$ and

$C \equiv (r\cos\theta_3, r\sin\theta_3)$

In triangle, we know that the circumcentre (O), centroid (G) and orthocentre (H) are collinear.

Using this result,

Slope of OH = Slope of GO

$$\Rightarrow \quad \frac{y'-0}{x'-0} = \frac{(\text{y cordinate of G})-0}{(\text{x cordinate of G})-0}$$

$$\Rightarrow \quad \frac{y'}{x'} = \frac{\sin\theta_1+\sin\theta_2+\sin\theta_3}{\cos\theta_1+\cos\theta_2+\cos\theta_3} \quad \text{Hence proved}$$

Example : 19

Find the coordinates of the points at unit distance from the lines : $3x - 4y + 1 = 0$, $8x + 6y + 1 = 0$

Solution

Let $L_1 \equiv 3x - 4y + 1 = 0$ and $L_2 \equiv 8x + 6y + 1 = 0$

In diagram, A, B, C and D are four points which lie at a unit distance from the two lines. You can also observe that A, B, C and D lie on angle bisectors of L_1 and L_2 .

Let (h, k) be the coordinates of a point of unit distance from each of the given lines.

$$\Rightarrow \quad \frac{|3h-4k+1|}{\sqrt{3^2+4^2}} \quad \text{and} \quad \frac{|8h+6k+1|}{\sqrt{8^2+6^2}}$$

$\Rightarrow \quad 3h - 4k + 1 = \pm 5$ and $8h + 6k + 1 = \pm 10$

$\Rightarrow \quad 3h - 4k - 4 = 0$ ....(i)

$3h - 4k + 6 = 0$ ....(ii)

$8h + 6k - 9 = 0$ ....(iii)

and $8h + 6k + 11 = 0$ ....(iv)

Solve (i) and (iii) to get : $(h, k) \equiv \left(\frac{6}{5}, \frac{-1}{10}\right)$

Solve (i) and (iv) to get : $(h, k) \equiv \left(\frac{-2}{5}, \frac{-13}{10}\right)$

Solve (ii) and (iii) to get : $(h, k) \equiv \left(0, \frac{3}{2}\right)$

Solve (ii) and (iv) to get : $(h, k) \equiv \left(\frac{-8}{5}, \frac{3}{10}\right)$

Hence the required four points are $\left(\frac{6}{5}, \frac{-1}{10}\right)$, $\left(\frac{-2}{5}, \frac{-13}{10}\right)$, $\left(0, \frac{3}{2}\right)$ and $\left(\frac{-8}{5}, \frac{3}{10}\right)$

Example : 20

Show that the area of the parallelogram formed by the line $3y - 2x = a$; $2y - 3x + a = 0$; $2x - 3y + 3a = 0$ and $3x - 2y = 2a$ is $\left(\frac{2a^2}{5}\right)$

Solution

The equations of four sides of the line are :

$2x - 3y + a = 0$ ....(i)

$-3x + 2y + a = 0$ ....(ii)

$2x - 3y + 3a = 0$ ....(iii)

$-3x + 2y + 2a = 0$ ....(iv)

Area of the parallelogram formed by above sides = $\frac{p_1p_2}{\sin\theta}$ ....(v)

where p_1 = perpendicular distance between parallel sides (i) and (iii),
p_2 = perpendicular distance between parallel sides (ii) and (iv),
θ = angle between adjacent sides (i) and (ii)

Find p_1

p_1 = perpendicular distance between (i) and (iii) = $\frac{|a-3a|}{\sqrt{2^2+(-3)^2}} = \frac{|2a|}{\sqrt{13}}$

Find p_2

p_2 = perpendicular distance between (ii) and (iv) = $\frac{|a-2a|}{\sqrt{2^2+(-3)^2}} = \frac{|a|}{\sqrt{13}}$

Find $\sin\theta$

If θ is the angle between (i) and (ii), then

$$\tan\theta = \frac{m_1-m_2}{1+m_1m_2} = \left|\frac{2/3-3/2}{1+(2/3).(3/2)}\right|$$

$\Rightarrow$ $\tan\theta = 5/12$

$\Rightarrow$ $\sin\theta = 5/13$

On substituting the values of p_1, p_2 and $\sin\theta$ in (v), we get

Area of the parallelogram formed by above sides = $\frac{\frac{|2a|}{\sqrt{13}} - \frac{|a|}{\sqrt{13}}}{5/13}$

$\Rightarrow$ Area of parallelogram = $2a^2$ /5q. units

Example : 21

The line joining the points A(2, 0); B(3, 1) is rotated about A in the anticlockwise direction through an angle of 15º. Find the equation of the line in the new position. If B goes to C in the new position, what will be the co-ordinate of C?

Solution

Slope of AB = $\frac{1-0}{3-2}$ = 1 = tan 45º

$\Rightarrow$ $\angle$BAX = 45º

Now line AB is rotated through an angle of 15º

$\Rightarrow$ $\angle$CAX = 60º and

AC = AB $\Rightarrow$ AC = $\sqrt{2}$

Equation of line AC in parametric form is :

$x = 2 + r\cos 60^\circ$

$y = 0 + r\sin 60^\circ$ ....(i)

Since AC = $\sqrt{2}$, pur r = $\sqrt{2}$ in (i) to get the coordinates of point C, i.e.

coordinates of C are $\left(\frac{4+\sqrt{2}}{2}, \frac{\sqrt{6}}{2}\right)$

Example : 22

Prove that two of the straight lines represented by the equation $ax^3 + bx^2y + cxy^2 + dy^3 = 0$ will be at right angles, if $a^2 + ac + bd + d^2 = 0$.

Solution

$ax^3 + bx^2y + cxy^2 + dy^3 = 0$ ....(i)

Equation (i) is a homogeneous equation of third degree in x and y

$\Rightarrow$ It represents combined equations of three straight lines passing through origin

Divide (i) by x^3 $\Rightarrow$ $a + b\,(y/x) + c\,(y/x)^2 + d\,(y/x)^3 = 0$

Put $(y/x) = m$

$\Rightarrow$ $a + bm + cm^2 + dm^3 = 0$

$\Rightarrow$ $dm^3 + cm^2 + bm + a = 0$

This is a cubic equation in 'm' with three roots m_1, m_2, m_3 [i.e. slopes of the three lines]

product of roots $= m_1 m_2 m_3 = -a/d$ ....(ii)

product of roots taken two at a time $= m_1 m_2 + m_2 m_3 + m_1 m_3 = b/d$ ....(iii)

sum of roots $= m_1 + m_2 + m_3 = -c/d$ ....(iv)

If any two lines are perpendicular to each other, then :

$m_1 m_2 = -1$ ....(v)

Solving (ii) and (v), we get

$m_3 = a/d$

On substituting the value of m_3 in (iv), we get

$m_1 + m_2 = -(a + c)/d$ ....(vi)

Solve (v) and (iii) and substitute the value of m_3 to get :

$m_3\,(m_1 + m_2) = (b + d)/d$

On substituting the value of $m_1 + m_2$ from (vi) in above equation, we get

$(a/d)\,[-(a + c)/d] = (b + d)/d$

$\Rightarrow$ $-a^2 - ac = bd + d^2$

$\Rightarrow$ $a^2 + ac + bd + d^2 = 0$

Hence proved

Example : 23

The sides of a triangle are, $L_r \equiv x \cos \theta_r + y \sin \theta_r - a_r = 0$, r = 1, 2, 3. Show that the orthocentre of the triangle is given by : L1 $\cos(\theta_2 - \theta_3) = L_2 \cos(\theta_3 - \theta_1) = L_3 \cos(\theta_1 - \theta_2)$.

Solution

Equation of any line through the point of intersection of $L_1 = 0$ and $L_2 = 0$ is

$L_1 + kL_2 = 0$, where k is a parameter.

$\Rightarrow$ $(\cos \theta_1 + k \cos \theta_2)\,x + (\sin \theta_1 + k \sin \theta_2)\,y - (a_1 + k\,a_2) = 0$ ....(i)

Line (i) will be perpendicular to $L_3 \equiv x \cos q_3 + y \sin \theta_3 - a_3 = 0$ if

[slope of (i)] × [slope of L_3] = – 1

$-[(\cos \theta_1 + k \cos\theta_2) / (\sin \theta_1 + k \sin \theta_2)]\,.\,[-(\cos \theta_3) / (\sin \theta_3)] = -1$

$\Rightarrow$ $k = -[\cos(\theta_3 - \theta_1)] / [\cos(\theta_2 - \theta_3)]$

On substituting the value of k in (i), we get the equation of one altitude as :

$L_1 \cos(\theta_2 - \theta_3) = L_2 \cos(\theta_3 - \theta_1)$ ....(ii)

Similarly, we can obtain the equations of second altitudes as :

$L_2 \cos(\theta_3 - \theta_1) = L_3 \cos(\theta_1 - \theta_2)$ ....(iii)

Solving the equations of altitudes (ii) and (iii), the orthocentre of the triangle is given by,

$L_1 \cos(\theta_2 - \theta_3) = L_2 \cos(\theta_3 - \theta_1) = L_3 \cos(\theta_1 - \theta_2)$.