

**Example : 1**

Find the maximum and minimum value of :

- (i)  $\sin \theta + \cos \theta$
- (ii)  $\sqrt{3} \sin \theta - \cos \theta$
- (iii)  $5 \sin \theta + 12 \cos \theta + 7$

**Solution**

Given expressions are in the form of  $a \sin \theta + b \cos \theta$

Express this in terms of one t-ratio by dividing and multiplying by  $\sqrt{a^2 + b^2}$

$$(i) \quad \sin \theta + \cos \theta = 1 \cdot \sin \theta + 1 \cdot \cos \theta$$

$$= \sqrt{2} \left( \frac{1}{\sqrt{2}} \sin \theta + \frac{1}{\sqrt{2}} \cos \theta \right)$$

$$= \sqrt{2} \left( \sin \theta \cos \frac{\pi}{4} + \cos \theta \sin \frac{\pi}{4} \right)$$

$$= \sqrt{2} \sin \left( \theta + \frac{\pi}{4} \right)$$

Now sine of angle must be between  $-1$  and  $1$

$$\Rightarrow -1 \leq \sin \left( \theta + \frac{\pi}{4} \right) \leq 1$$

$$\Rightarrow -\sqrt{2} \leq \sqrt{2} \sin \left( \theta + \frac{\pi}{4} \right) \leq \sqrt{2}$$

So maximum value of  $\sin \theta + \cos \theta$  is  $\sqrt{2}$  and minimum value of  $\sin \theta + \cos \theta$  is  $-\sqrt{2}$

$$(ii) \quad \sqrt{3} \sin \theta - \cos \theta = 2 \left( \frac{\sqrt{3}}{2} \sin \theta - \frac{1}{2} \cos \theta \right)$$

$$= 2 \left( \sin \theta \cos \frac{\pi}{6} - \cos \theta \sin \frac{\pi}{6} \right)$$

$$= 2 \sin \left( \theta - \frac{\pi}{6} \right)$$

$$\text{as } -1 \therefore \sin \left( \theta - \frac{\pi}{6} \right) \leq 1$$

$$\Rightarrow -2 \leq 2 \sin \left( \theta - \frac{\pi}{6} \right) \leq 2$$

so maximum value is  $2$  and minimum value is  $-2$

- (iii) Consider  $5 \sin \theta + 12 \cos \theta = 13$  [ $5/13 \sin \theta + 12/13 \cos \theta$ ]  
construct a triangle with sides,  $5, 12, 13$ . If  $\alpha$  is an angle of triangle,  
then  $\cos \alpha = 5/13, \sin \alpha = 12/13,$

$$\Rightarrow 5 \sin \theta + 12 \cos \theta = 13 [\sin \theta \cos \alpha + \cos \theta \sin \alpha]$$

$$5 \sin \theta + 12 \cos \theta + 7 = 13 [\sin (\theta + \alpha)] + 7$$

$$\text{as } -1 \leq \sin (\theta + \alpha) \leq 1$$

$$\Rightarrow -13 \leq 13 \sin (\theta + \alpha) \leq 13$$

$$-13 + 7 \leq 13 \sin (\theta + \alpha) + 7 \leq 13 + 7$$

So maximum value is  $20$  and minimum value is  $-6$ .

**Example : 2**

Show that  $\sin \pi/13$  is a root of  $8x^3 - 4x^2 - 4x + 1 = 0$

**Solution**

Let  $\theta = \pi/14$

$$\begin{aligned} \Rightarrow 4\theta &= \pi/2 - 3\theta \\ \Rightarrow \sin 4\theta &= \sin [\pi/2 - 2\theta] = \cos 3\theta \\ \Rightarrow 2[2 \sin \theta \cos \theta] \cos 2\theta &= \cos \theta [4 \cos^2\theta - 3] \\ \Rightarrow 4 \sin \theta [1 - 2 \sin^2\theta] &= 4 - 4 \sin^2\theta - 3 \\ \Rightarrow 8 \sin^3\theta - 4 \sin^2\theta - 4 \sin \theta + 1 &= 0 \\ \Rightarrow \sin \theta &\text{ is root of } 8x^3 - 4x^2 - 4x + 1 = 0 \end{aligned}$$

**Example : 3**

If  $\alpha$  and  $\beta$  are roots of  $a \tan \theta + b \sec \theta = c$ , find the value of :

(i)  $\tan [\alpha + \beta]$       (ii)  $\cos [\alpha + \beta]$

**Solution**

(i) To find  $\tan (\alpha + \beta)$ , we need  $\tan \alpha + \tan \beta$  and  $\tan \alpha \tan \beta$ , so express the given equation in terms of a quadratic in  $\tan \theta$  where sum of roots is  $\tan \alpha + \tan \beta$  and product of roots is  $\tan \alpha \tan \beta$

Consider  $a \tan \theta + b \sec \theta = c$

$$\begin{aligned} \Rightarrow (c - a \tan \theta)^2 &= b^2 \sec^2\theta \\ \Rightarrow c^2 + a^2 \tan^2\theta - 2ac \tan \theta &= b^2 + b^2 \tan^2\theta \\ \Rightarrow (a^2 - b^2) \tan^2\theta - 2ac \tan \theta + c^2 - b^2 &= 0 \end{aligned}$$

$$\tan \alpha + \tan \beta = \text{sum of roots} = \frac{2ac}{a^2 - b^2}$$

$$\tan \alpha \tan \beta = \text{product of roots} = \frac{c^2 - b^2}{a^2 - b^2}$$

$$\Rightarrow \tan (\alpha + \beta) = \frac{\tan \alpha + \tan \beta}{1 - \tan \alpha \tan \beta} = \frac{\frac{2ac}{a^2 - b^2}}{1 - \frac{c^2 - b^2}{a^2 - b^2}} = \frac{2ac}{a^2 - c^2}$$

(ii) To find  $\cos (\alpha + \beta)$ , express given equation as :

1. quadratic in  $\cos \theta$  to find  $\cos \alpha \cos \beta$

2. quadratic in  $\sin \theta$  to find  $\sin \alpha \sin \beta$

Given equation can be written as :

$$\begin{aligned} a \sin \theta + b &= c \cos \theta \\ \Rightarrow a^2 \sin^2\theta + b^2 + 2ab \sin \theta &= c^2 \cos^2\theta = c^2 (1 - \sin^2\theta) \\ \Rightarrow (a^2 + c^2) \sin^2\theta + 2ab \sin \theta + b^2 - c^2 &= 0 \end{aligned}$$

$$\text{Hence product of roots of } \sin \alpha \sin \beta = \frac{b^2 - c^2}{a^2 + c^2}$$

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$$\text{Hence product of roots of } \sin \alpha \sin \beta = \frac{b^2 - c^2}{a^2 + c^2}$$

Given equation can be written as

$$\begin{aligned} a \sin \theta + b &= c \cos \theta \\ \Rightarrow a^2 \sin^2\theta &= (c \cos \theta - b)^2 \\ \Rightarrow a^2 (1 - \cos^2\theta) &= c^2 \cos^2\theta + b^2 - 2bc \cos \theta \\ \Rightarrow (a^2 + c^2) \cos^2 \theta - 2bc \cos \theta + b^2 - a^2 &= 0 \end{aligned}$$

$$\text{so product of roots} = \cos \alpha \cos \beta = \frac{b^2 - a^2}{a^2 + c^2}$$

Now  $\cos(\alpha + \beta) = \cos \alpha \cos \beta - \sin \alpha \sin \beta$

$$= \frac{b^2 - a^2}{a^2 + c^2} - \frac{b^2 - c^2}{a^2 + c^2} = \frac{c^2 - a^2}{a^2 + c^2}$$

**Example : 4**

If  $m \tan(\theta - 30^\circ) = n \tan(\theta + 120^\circ)$ , then show that :  $\cos 2\theta = \frac{m+n}{2(m-n)}$

**Solution**

$$\frac{\tan(\theta + 120^\circ)}{\tan(\theta - 30^\circ)} = \frac{m}{n}$$

$$\Rightarrow \frac{\sin(\theta + 120^\circ)\cos(\theta - 30^\circ)}{\cos(\theta + 120^\circ)\sin(\theta - 30^\circ)} = \frac{m}{n}$$

$$\Rightarrow \frac{\sin(\theta + 120^\circ)\cos(\theta - 30^\circ) - \cos(\theta + 120^\circ)\sin(\theta - 30^\circ)}{\sin(\theta + 120^\circ)\cos(\theta - 30^\circ) + \cos(\theta + 120^\circ)\sin(\theta - 30^\circ)} = \frac{m-n}{m+n}$$

$$\Rightarrow \frac{\sin 150^\circ}{\sin(2\theta + 90^\circ)} = \frac{m-n}{m+n}$$

$$\Rightarrow \frac{\sin[(\theta + 120^\circ) - (\theta - 30^\circ)]}{\sin[(\theta + 120^\circ) + (\theta - 30^\circ)]} = \frac{m-n}{m+n}$$

$$\Rightarrow \frac{1}{2} (m+n) = (m-n) \cos 2\theta$$

$$\Rightarrow \cos 2\theta = \frac{m+n}{2(m-n)}$$

**Example : 5**

Show that :  $\sin^2 B = \sin^2 A + \sin^2(A - B) - 2 \sin A \cos B \sin(A - B)$

**Solution**

Starting from RHS :

$$\begin{aligned} \text{RHS} &= \sin^2 A + \sin^2(A - B) - 2 \sin A \cos B \sin(A - B) \\ &= \sin^2 A + \sin^2(A - B) - [\sin(A + B) + \sin(A - B)] \sin(A - B) \\ &= \sin^2 A + \sin^2(A - B) - \sin(A + B) \sin(A - B) - \sin^2(A - B) \\ &= \sin^2 A - [\sin^2 A - \sin^2 B] \\ &= \sin^2 B = \text{LHS} \end{aligned}$$

**Example : 6**

If  $\tan \frac{\theta}{2} = \sqrt{\frac{1-e}{1+e}} \tan \frac{\phi}{2}$ , show that :  $\cos \phi = \frac{\cos \theta - e}{1 - e \cos \theta}$

**Solution**

We have to find  $\cos \phi$  in terms of  $e$  and  $\cos \theta$ , so try to convert  $\tan \theta/2$  to  $\cos \phi$

$$\tan^2 \frac{\theta}{2} = \frac{1-e}{1+e} \tan^2 \frac{\phi}{2}$$

$$\Rightarrow \tan^2 \frac{\phi}{2} = \frac{1+e}{1-e} \tan^2 \frac{\theta}{2} = \frac{1+e}{1-e} \left( \frac{1-\cos \theta}{1+\cos \theta} \right)$$

$$\Rightarrow \frac{\tan^2 \frac{\phi}{2}}{1} = \frac{1+e - \cos \theta - e \cos \theta}{1-e + \cos \theta - e \cos \theta}$$

$$\Rightarrow \frac{1 - \tan^2 \phi / 2}{1 + \tan^2 \phi / 2} = \frac{(1 + e + \cos \theta - e \cos \theta) - (1 + e \cos \theta - e \cos \theta)}{(1 - e + \cos \theta - e \cos \theta) + (1 + e - \cos \theta - e \cos \theta)}$$

$$\Rightarrow \cos \phi = \frac{-2e + 2 \cos \theta}{2 - 2e \cos \theta} = \frac{\cos \theta - e}{1 - e \cos \theta}$$

**Example : 7**

If  $\tan \beta = \frac{\tan \alpha + \tan \gamma}{1 + \tan \alpha \tan \gamma}$ , prove that :  $\sin 2\beta = \frac{\sin 2\alpha + \sin 2\gamma}{1 + \sin 2\alpha \sin 2\gamma}$

**Solution**

We are given  $\tan \beta$  in terms of  $\alpha$  and  $\gamma$ , so we have to express  $\sin 2\beta$  in terms of  $\alpha, \gamma$ . Hence we will start with  $\sin 2\beta = (2 \tan \beta) / (1 + \tan^2 \beta)$  and substitute for  $\tan \beta$  in RHS. Also, as the final expression does not contain  $\tan \alpha$  and  $\tan \gamma$ , so express  $\tan \beta$  in terms of sine and cosine.

$$\tan \beta = \frac{\sin \alpha \cos \gamma + \cos \gamma \sin \alpha}{\cos \alpha \cos \gamma + \sin \alpha \sin \gamma} = \frac{\sin(\alpha + \gamma)}{\cos(\alpha - \gamma)}$$

$$\text{Now } \sin \beta = \frac{2 \tan \theta}{1 + \tan^2 \beta}$$

$$\Rightarrow \sin 2\beta = \frac{2 \frac{\sin(\alpha + \gamma)}{\cos(\alpha - \gamma)}}{1 + \frac{\sin^2(\alpha + \gamma)}{\cos^2(\alpha - \gamma)}} = \frac{2 \sin(\alpha + \gamma) \cos(\alpha - \gamma)}{\cos^2(\alpha - \gamma) + \sin^2(\alpha + \gamma)}$$

$$= \frac{\sin[\alpha + \gamma + \alpha - \gamma] + \sin[\alpha + \gamma - \alpha - \gamma]}{1 + \sin^2(\alpha + \gamma) - \sin^2(\alpha - \gamma)}$$

$$= \frac{\sin 2\alpha + \sin 2\gamma}{1 + \sin[\alpha + \gamma + \alpha - \gamma] \sin[\alpha + \gamma - \alpha - \gamma]}$$

$$\Rightarrow \sin 2\beta = \frac{\sin 2\alpha + \sin 2\gamma}{1 + \sin 2\alpha \sin 2\gamma}$$

**Example : 8**

If  $2 \tan \alpha = 3 \tan \beta$ , then show that :  $\tan(\alpha - \beta) = \frac{\sin 2\beta}{5 - \cos 2\beta}$

**Solution**

We have to express  $\tan(\alpha - \beta)$  in terms of  $\beta$  only. Starting with standard result of  $\tan(\alpha - \beta)$  and substituting for  $\tan \alpha = 3/2 \tan \beta$  in RHS, we have :

$$\Rightarrow \tan(\alpha - \beta) = \frac{\tan \alpha + \tan \beta}{1 + \tan \alpha \tan \beta} = \frac{3/2 \tan \beta - \tan \beta}{1 + 3/2 \tan^2 \beta}$$

$$\Rightarrow \tan(\alpha - \beta) = \frac{\tan \beta}{2 + 3 \tan^2 \beta} = \frac{\sin \beta \cos \beta}{2 \cos^2 \beta + 3 \sin^2 \beta} = \frac{2 \sin \beta \cos \beta}{4 \cos^2 \beta + 6 \sin^2 \beta} = \frac{\sin 2\beta}{2(1 + \cos 2\beta) + 3(1 - \cos 2\beta)}$$

$$\Rightarrow \tan(\alpha - \beta) = \frac{\sin 2\beta}{5 - \cos 2\beta}$$

**Example : 9**

Prove that  $\tan \alpha + 2 \tan 2\alpha + 4 \tan 4\alpha + 8 \cot 8\alpha = \cot \alpha$ .

**Solution**

For this problem, use the result  $\cos \alpha - \tan \alpha = 2 \cot 2\alpha$

Now we can express the above relation as :

$$\tan \alpha = \cot \alpha - 2 \cot 2\alpha$$

Replacing  $\alpha$  by  $2\alpha$  :

$$\tan 2\alpha = \cot 2\alpha - 2 \cot 4\alpha$$

Replacing  $\alpha$  by  $4\alpha$  :

$$\tan 4\alpha = \cot 4\alpha - 2 \cot 8\alpha$$

Multiplying these equations by 1, 2, 4 respectively and adding them together, we get

$$\tan \alpha + 2 \tan 2\alpha + 4 \tan 4\alpha = \cot \alpha - 8 \cot 8\alpha$$

$$\Rightarrow \tan \alpha + 2 \tan 2\alpha + 4 \tan 4\alpha + 8 \cot 8\alpha = \cot \alpha$$

**Example : 10**

Show that  $\cos \frac{\pi}{33} \cos \frac{2\pi}{33} \cos \frac{4\pi}{33} \cos \frac{8\pi}{33} \cos \frac{16\pi}{33} = \frac{1}{32}$

**Solution**

If  $\theta = \pi/33$ , observe that pattern  $\cos \theta \cos 2\theta \cos 4\theta \cos 8\theta$

In this pattern  $\sin 2\alpha = 2 \sin \alpha \cos \alpha$  will be used repeatedly in LHS, so multiply and divide by  $2 \sin \pi/33$ .

$$\text{LHS} = \frac{\left(2 \sin \frac{\pi}{33} \cos \frac{\pi}{33}\right) \left(\cos \frac{2\pi}{33} \cos \frac{4\pi}{33} \cos \frac{8\pi}{33} \cos \frac{16\pi}{33}\right)}{2 \sin \frac{\pi}{33}}$$

$$= \frac{2 \left(\sin \frac{2\pi}{33} \cos \frac{2\pi}{33}\right) \left(\cos \frac{4\pi}{33} \cos \frac{8\pi}{33} \cos \frac{16\pi}{33}\right)}{2.2 \sin \frac{\pi}{33}}$$

..... following the same pattern .... we have

$$\text{LHS} = \frac{\sin \frac{32\pi}{33}}{2^5 \cdot \sin \frac{\pi}{33}} = \frac{\sin \left(\pi - \frac{\pi}{33}\right)}{32 \sin \frac{\pi}{33}} = \frac{1}{32}$$

**Example : 11**

Show that  $\tan 6^\circ \sin 42^\circ \sin 66^\circ \sin 78^\circ = 1/16$

**Solution**

Note that  $(66 + 6)/2 = 36^\circ$   $(66 - 6)/2 = 30^\circ$ . Hence  $\sin 6^\circ$  and  $\sin 66^\circ$  should be combined.

$$\begin{aligned} \text{LHS} &= 1/4 [2 \sin 6^\circ \sin 66^\circ] [2 \sin 42^\circ \sin 78^\circ] \\ &= 1/4 [\cos (6^\circ + 66^\circ) - \cos (6^\circ - 66^\circ)] [\cos (42^\circ - 78^\circ) - \cos (42^\circ + 78^\circ)] \\ &= 1/4 [\cos 60^\circ - \cos 72^\circ] [\cos 36^\circ - \cos 120^\circ] \end{aligned}$$

Substituting the values, we get

$$\begin{aligned} \text{LHS} &= \frac{1}{4} \left(\frac{1}{2} - \frac{\sqrt{5}-1}{4}\right) \left(\frac{\sqrt{5}+1}{4} + \frac{1}{2}\right) = \frac{1}{4} \left(\frac{2-\sqrt{5}+1}{4}\right) \left(\frac{\sqrt{5}+1+2}{4}\right) \\ &= \frac{1}{64} (3 - \sqrt{5}) (3 + \sqrt{5}) = \frac{1}{16} \text{ RHS} \end{aligned}$$

**Example : 12**

If  $\alpha = 2\pi/7$ , show that  $\tan \alpha \tan 2\alpha \tan 4\alpha \tan \alpha = -7$

**Solution**

$$\text{LHS} = \frac{\sin \alpha \sin 2\alpha \cos 4\alpha + \sin 2\alpha \sin 4\alpha \cos \alpha + \sin 4\alpha \sin \alpha \cos 2\alpha}{\cos \alpha \cos 2\alpha \cos 4\alpha}$$

We will use the formula for  $\cos (A + B + C)$

$\cos (A + B + C) = \cos A \cos B \cos C - \sin A \sin B \cos C - \sin B \sin C \cos A - \sin A \cos B$

$$\begin{aligned} \Rightarrow \text{LHS} &= \frac{\cos \alpha \cos 2\alpha \cos 4\alpha (\alpha + 2\alpha + 4\alpha)}{\cos \alpha \cos 2\alpha \cos 4\alpha} \\ &= 1 - \frac{\cos 7\alpha}{\cos \alpha \cos 2\alpha \cos 4\alpha} = 1 - \frac{\cos 2\pi(2 \sin \alpha)}{2 \sin \alpha \cos \alpha \cos 2\alpha \cos 4\alpha} \\ &= 1 - \frac{4 \sin \alpha}{2 \sin \alpha \cos 2\alpha \cos 4\alpha} \\ &= 1 - \frac{8 \sin \alpha}{2 \sin 4\alpha \cos 4\alpha} \\ &= 1 - \frac{8 \sin \alpha}{\sin 8\alpha} = 1 - \frac{8 \sin \alpha}{\sin(2\pi + \alpha)} = 1 - \frac{8 \sin \alpha}{\sin \alpha} = -7 \end{aligned}$$

**Example : 13**

Show that  $\cos 2\pi/7 + \cos 4\pi/7 + \cos 6\pi/7 = -1/2$ .

**Solution**

$$\begin{aligned} \text{LHS} &= \frac{2 \sin \frac{\pi}{7} \left( \cos \frac{2\pi}{7} + \cos \frac{4\pi}{7} + \cos \frac{6\pi}{7} \right)}{2 \sin \frac{\pi}{7}} \\ &= \frac{1}{2 \sin \frac{\pi}{7}} \left[ \left( \sin \frac{3\pi}{7} - \sin \frac{\pi}{7} \right) + \left( \sin \frac{5\pi}{7} - \sin \frac{3\pi}{7} \right) + \left( \sin \frac{7\pi}{7} - \sin \frac{5\pi}{7} \right) \right] \\ &= \frac{\sin \pi - \sin \frac{\pi}{7}}{2 \sin \frac{\pi}{7}} = -\frac{1}{2} \end{aligned}$$

**Alternative Method :**

We can also use the relation :

$$\begin{aligned} \cos a + \cos (a + d) + \dots + \cos (a + (n-1)d) &= \frac{\sin nd/2}{\sin d/2} \cos \left( \frac{2a + (n-1)d}{2} \right) \\ \Rightarrow \text{LHS} &= \frac{\sin 3 \left( \frac{2\pi/7}{2} \right)}{\sin \frac{2\pi/7}{2}} \cos \left( \frac{\frac{4\pi}{7} + 2 \left( \frac{2\pi}{7} \right)}{2} \right) = \frac{\sin 3\pi/7}{\sin \pi/7} \cos \left( \frac{4\pi}{7} \right) = \frac{\sin \pi + \sin(-\pi/7)}{2 \sin \pi/7} = -\frac{1}{2} \end{aligned}$$

**Example : 14**

Let  $\cos \alpha \cos \beta \cos \phi = \cos \gamma \cos \theta$  and  $\sin \alpha = 2 \sin \frac{\phi}{2} \sin \frac{\theta}{2}$ , then prove that  $\tan^3 \frac{\alpha}{2} = \tan^2 \frac{\beta}{2} \tan^2 \frac{\gamma}{2}$ .

**Solution**

From the given three equations, we have to eliminate two variables,  $\theta$  and  $\phi$

$$\cos \alpha = \cos \beta \cos \phi = \cos \gamma \cos \theta$$

$$\Rightarrow \cos \phi = \frac{\cos \alpha}{\cos \beta} ; \cos \theta = \frac{\cos \alpha}{\cos \gamma}$$

$$\Rightarrow 2 \sin^2 \frac{\phi}{2} = 1 - \frac{\cos \alpha}{\cos \beta} ; 2 \sin^2 \frac{\theta}{2} = 1 - \frac{\cos \alpha}{\cos \gamma}$$

substitute these is  $\sin \alpha = 2 \sin \frac{\phi}{2} \sin \frac{\theta}{2}$

$$\Rightarrow \sin \alpha = \sqrt{\left(1 - \frac{\cos \alpha}{\cos \beta}\right) \left(1 - \frac{\cos \alpha}{\cos \gamma}\right)}$$

$$\Rightarrow \sin^2 \alpha \left(1 - \frac{\cos \alpha}{\cos \beta} - \frac{\cos \alpha}{\cos \gamma} + \frac{\cos^2 \alpha}{\cos \beta \cos \gamma}\right)$$

$$\Rightarrow \cos \alpha \left(1 + \frac{1}{\cos \beta \cos \gamma}\right) = \frac{\cos \beta + \cos \gamma}{\cos \beta \cos \gamma}$$

$$\Rightarrow \cos \alpha (\cos \beta \cos \alpha + 1) = \cos \beta + \cos \gamma$$

$$\Rightarrow \cos \alpha = \frac{\cos \beta \cos \gamma}{1 + \cos \beta \cos \gamma}$$

Using component and dividendo, we get :

$$\frac{1 - \cos \alpha}{1 + \cos \alpha} = \frac{1 + \cos \beta \cos \gamma - \cos \beta - \cos \gamma}{1 + \cos \beta \cos \gamma + \cos \beta + \cos \gamma}$$

$$\Rightarrow \tan^2 \frac{\alpha}{2} = \frac{(1 - \cos \beta)(1 - \cos \gamma)}{(1 + \cos \beta)(1 + \cos \gamma)}$$

$$\Rightarrow \tan^2 \frac{\alpha}{2} = \tan^2 \frac{\beta}{2} \tan^2 \frac{\gamma}{2}$$

**Example : 15**

If  $\frac{\sin^4 \alpha}{a} + \frac{\cos^4 \alpha}{b} = \frac{1}{a+b}$ , then show that :  $\frac{\sin^8 \alpha}{a^3} + \frac{\cos^8 \alpha}{b^3} = \frac{1}{(a+b)^3}$

**Solution**

Express the given equation in quadratic in terms of  $\sin^2 \alpha$ .

$$\frac{\sin^4 \alpha}{a} + \frac{\cos^4 \alpha}{b} = \frac{1}{a+b}$$

$$\Rightarrow \frac{\sin^4 \alpha}{a} + \frac{(\sin^2 \alpha)^2}{b} = \frac{1}{a+b}$$

$$\Rightarrow (a+b)^2 \sin^4 \alpha - 2a(a+b) \sin^2 \alpha + a^2 = 0$$

$$\Rightarrow [(a+b) \sin^2 \alpha - a]^2 = 0$$

$$\Rightarrow \sin^2 \alpha = \frac{a}{a+b}$$

$$\Rightarrow \cos^2 \alpha = \frac{b}{a+b}$$

Now LHS of the equation to be proved is :

$$\begin{aligned}
 &= \frac{\sin^8 \alpha}{a^3} + \frac{\cos^8 \alpha}{b^3} \\
 &= \frac{a^4}{a^3(a+b)^4} + \frac{b^4}{b^3(a+b)^4} \\
 &= \frac{a+b}{(a+b)^4} = \frac{1}{(a+b)^3} = \text{RHS}
 \end{aligned}$$

**Example : 16**

If  $\frac{\cos^4 x}{\cos^2 y} = \frac{\sin^4 x}{\sin^2 y} = 1$ , then prove that :  $\frac{\cos^4 y}{\cos^2 x} = \frac{\sin^4 y}{\sin^2 x} = 1$

**Solution**

Consider  $\frac{\cos^4 x}{\cos^2 y} = \frac{\sin^4 x}{\sin^2 y} = 1$

$$\Rightarrow \left( \frac{\cos^4 x}{\cos^2 y} - \cos^2 x \right) + \left( \frac{\sin^4 x}{\sin^2 y} - \sin^2 x \right) = 0$$

$$\Rightarrow \frac{\cos^2 x}{\cos^2 y} (\cos^2 x - \cos^2 y) + \frac{\sin^2 x}{\sin^2 y} (\sin^2 x - \sin^2 y) = 0$$

$$\Rightarrow \frac{\cos^2 x}{\cos^2 y} (\cos^2 x - \cos^2 y) + \frac{\sin^2 x}{\sin^2 y} (\cos^2 x - \cos^2 y) = 0$$

$$\Rightarrow (\cos^2 x - \cos^2 y) \left[ \frac{\cos^2 x}{\cos^2 y} - \frac{\sin^2 x}{\sin^2 y} \right] = 0$$

$$\Rightarrow \cos^2 x = \cos^2 y \text{ or } \tan^2 x = \tan^2 y \dots\dots\dots(i)$$

$$= \frac{\cos^4 y}{\cos^2 x} + \frac{\sin^4 y}{\sin^2 x}$$

$$= \frac{\cos^4 x}{\cos^2 x} + \frac{\sin^4 x}{\sin^2 x} \quad (\text{using i})$$

$$= \cos^2 x + \sin^2 x = 1 = \text{RHS}$$

**Example : 17**

Show that  $1 + \sin^2 \alpha + \sin^2 \beta > \sin \alpha + \sin \beta + \sin \alpha \sin \beta$

**Solution**

Consider the expression :

$$\begin{aligned}
 &a^2 + b^2 + c^2 - ab - bc - ca \\
 &1/2 [(a - b)^2 + (b - c)^2 + (c - a)^2]
 \end{aligned}$$

which is positive

$$\Rightarrow (a^2 + b^2 + c^2 - ab - bc - ca) > 0 \text{ If } a, b, c \text{ are unequal}$$

Taking  $a = 1, b = \sin \alpha, c = \sin \beta$ , we get

$$1 + \sin^2 \alpha + \sin^2 \beta - \sin \alpha - \sin \alpha \sin \beta - \sin \beta > 0$$

$$\Rightarrow 1 + \sin^2 \alpha + \sin^2 \beta > \sin \alpha + \sin \beta + \sin \alpha \sin \beta$$

**Example : 18**

If  $\frac{ax}{\cos\theta} + \frac{by}{\sin\theta} = a^2 - b^2$  and  $\frac{ax \sin\theta}{\cos^2\theta} - \frac{by \cos\theta}{\sin^2\theta} = 0$ , then show that  $(ax)^{2/3} + (by)^{2/3} = (a^2 - b^2)^{2/3}$ .

**Solution**

$$\text{From } \frac{ax \sin\theta}{\cos^2\theta} - \frac{by \cos\theta}{\sin^2\theta} = 0 \Rightarrow \tan\theta = \left(\frac{by}{ax}\right)^{1/3}$$

Substituting this value of  $\tan\theta$  in the other given condition.

We have :  $ax \sec\theta + by \operatorname{cosec}\theta = a^2 - b^2$

$$\Rightarrow ax \sqrt{1 + \tan^2\theta} + by \sqrt{1 + \cot^2\theta} = a^2 - b^2$$

$$\Rightarrow ax \sqrt{1 + \left(\frac{by}{ax}\right)^{2/3}} + by \sqrt{1 + \left(\frac{ax}{by}\right)^{2/3}} = a^2 - b^2$$

$$\Rightarrow \frac{ax}{(ax)^{1/3}} \sqrt{(ax)^{2/3} + (by)^{2/3}} + \frac{by}{(by)^{1/3}} \sqrt{(ax)^{2/3} + (by)^{2/3}} = a^2 - b^2$$

$$\Rightarrow [(ax)^{2/3} + (by)^{2/3}] \sqrt{(ax)^{2/3} + (by)^{2/3}} = a^2 - b^2$$

$$\Rightarrow ((ax)^{2/3} + (by)^{2/3})^{3/2} = a^2 - b^2$$

$$\Rightarrow (ax)^{2/3} + (by)^{2/3} = (a^2 - b^2)^{2/3}$$

**Example : 19**

If  $a, b, c$  are unequal. Eliminate  $\theta$  from :  $a \cos\theta + b \sin\theta = c$  and  $a \cos^2\theta + 2a \sin\theta \cos\theta + b \sin^2\theta = c$ .

**Solution**

Consider  $a \cos\theta + b \sin\theta = c$

$$\Rightarrow a^2 \cos^2\theta + b^2 \sin^2\theta + 2ab \sin\theta \cos\theta = c^2$$

$$\Rightarrow (a^2 - c^2) \cos^2\theta + 2ab \sin\theta \cos\theta + (b^2 - c^2) \sin^2\theta = 0 \quad \dots\dots\dots(i)$$

Now consider  $a \cos^2\theta + 2ab \sin\theta \cos\theta + (b^2 - c^2) \sin^2\theta = c$

$$\Rightarrow (a - c) \cos^2\theta + 2a \sin\theta \cos\theta + (b - c) \sin^2\theta = 0 \quad \dots\dots\dots(ii)$$

Use cross-multiplication method on (i) and (ii).

$$(a^2 - c^2) \cos^2\theta + 2ab \sin\theta \cos\theta + (b^2 - c^2) \sin^2\theta = 0$$

$$(a - c) \cos^2\theta + 2a \sin\theta \cos\theta + (b - c) \sin^2\theta = 0$$

$$\Rightarrow \frac{\cos^2\theta}{2ab(b-c) - 2a(b^2 - c^2)} = \frac{-\sin\theta \cos\theta}{(b-c)(a^2 - c^2) - (a-c)(b^2 - c^2)} = \frac{\sin^2\theta}{2a(a^2 - c^2) - 2ab(a-c)}$$

$$\Rightarrow \frac{\cos^2\theta}{2ab(b-c)(-c)} = \frac{-\sin\theta \cos\theta}{(b-c)(a-c)(a-b)} = \frac{\sin^2\theta}{2a(a-c)(a+c-b)}$$

$$\Rightarrow (a-b)^2 (b-c)^2 (a-c)^2 = 4a^2c (b-c) (c-a) (a+c-b)$$

$$\Rightarrow (a-b)^2 (b-c) (c-a) = 4a^2c (a+c-b)$$

**Example : 20**

If  $m^2 + m'^2 + 2m = m' \cos\theta = 1$ ,  $n^2 + n'^2 + 2nn' \cos\theta = 1$  and  $mn + m' n' + (mm' + m'n) \cos\theta = 0$ , then prove that :  $m^2 + n^2 = \operatorname{cosec}^2\theta$

**Solution**

Consider the first given condition :

$$m^2 + m'^2 + 2mm' \cos\theta = 1$$

$$\Rightarrow m^2 (\sin^2\theta + \cos^2\theta) + m'^2 + 2mm' \cos\theta = 1$$

$$\Rightarrow m^2 \cos^2\theta + m'^2 + 2mm' \cos\theta = 1 - m^2 \sin^2\theta$$

$$\Rightarrow (m \cos\theta + m')^2 = 1 - m^2 \sin^2\theta \quad \dots\dots\dots(i)$$

Similarly using the second given condition, we can get

$$(n \cos\theta + n')^2 = 1 - n^2 \sin^2\theta \quad \dots\dots\dots(ii)$$

By multiplying (i) and (ii) we can prove the required relation.

$$(m \cos \theta + m')^2 (n \cos \theta + n')^2 = (1 - m^2 \sin^2 \theta) (1 - n^2 \sin^2 \theta)$$

$$\Rightarrow [mn \cos^2 \theta + m'n' + (m'n + mn') \cos \theta]^2 = 1 - m^2 \sin^2 \theta - n^2 \sin^2 \theta + m^2 n^2 \sin^4 \theta$$

Using the third given condition in LHS, we get :

$$[mn \cos^2 \theta - mn]^2 = 1 - m^2 \sin^2 \theta - n^2 \sin^2 \theta + m^2 n^2 \sin^4 \theta$$

$$m^2 n^2 \sin^4 \theta = 1 - \sin^2 \theta (m^2 + n^2) + m^2 n^2 \sin^4 \theta$$

$$\Rightarrow m^2 + n^2 = \operatorname{cosec}^2 \theta$$

### Example : 21

If  $\tan \theta = m/n$  and  $\theta = 3\phi$  ( $0 < \theta < \pi/2$ ) show that :  $\frac{m}{\sin \phi} - \frac{n}{\cos \phi} = 2\sqrt{m^2 + n^2}$

#### Solution

$$\tan \theta = m/n$$

$$\Rightarrow \sin \theta = \frac{m}{\sqrt{m^2 + n^2}} \text{ and } \cos \theta = \frac{n}{\sqrt{m^2 + n^2}}$$

$$\begin{aligned} \text{LHS of given equation} &= \frac{m}{\sin \phi} - \frac{n}{\cos \phi} = \sqrt{m^2 + n^2} \left( \frac{\sin \theta}{\sin \phi} - \frac{\cos \theta}{\cos \phi} \right) \\ &= \sqrt{m^2 + n^2} \left( \frac{\sin(\theta - \phi)}{\sin \phi \cos \phi} \right) = 2 \sqrt{m^2 + n^2} \frac{\sin(3\theta - \phi)}{2 \sin \phi \cos \phi} \\ &= 2 \sqrt{m^2 + n^2} \frac{\sin 2\phi}{2 \sin \phi \cos \phi} = 2 \sqrt{m^2 + n^2} = \text{RHS} \end{aligned}$$

### Example : 22

If  $A + B + C = \pi$ , then show that

- (i)  $\sin^2 A + \sin^2 B - \sin^2 C = 2 \sin A \sin B \cos C$
- (ii)  $\cos^2 A/2 + \cos^2 B/2 + \cos^2 C/2 = 2 + 2 \sin A/2 \sin B/2 \sin C/2$
- (iii)  $\sin^2 A + \sin^2 B + \sin^2 C = 2 + 2 \cos A \cos B \cos C$

#### Solution

- (i) Starting from LHS
 
$$\begin{aligned} &= \sin^2 A + \sin^2 B - \sin^2 C \\ &= \sin^2 A + \sin(B + C) \sin(B - C) \\ &= \sin^2 A + \sin(\pi - A) \sin(B - C) \\ &= \sin A [\sin A + \sin(B - C)] \\ &= \sin A [\sin(\pi - (B + C)) + \sin(B - C)] \\ &= \sin A [\sin(B + C) + \sin(B - C)] \\ &= \sin A [2 \sin B \cos C] = 2 \sin A \sin B \cos C = \text{RHS} \end{aligned}$$
- (ii) LHS =  $\cos^2 A/2 + (1 - \sin^2 B/2) + \cos^2 C/2$ 

$$\begin{aligned} &= 1 + (\cos^2 A/2 - \sin^2 B/2) + \cos^2 C/2 \\ &= 1 + \cos(A + B)/2 \cos(A - B)/2 + \cos^2 C/2 \\ &= 1 + \sin C/2 \cos(A - B)/2 + 1 - \sin^2 C/2 \\ &= 2 + \sin C/2 [\cos(A - B)/2 - \sin C/2] \\ &= 2 + \sin C/2 [\cos(A - B)/2 = \cos(A + B)/2] \\ &= 2 + 2 \sin C/2 \sin A/2 \sin B/2 = \text{RHS} \end{aligned}$$
- (iii) LHS =  $\sin^2 A + \sin^2 B + \sin^2 C$ 

$$\begin{aligned} &= 1 - (\cos^2 A - \sin^2 B) + \sin^2 C \\ &= 1 - \cos(A + B) \cos(A - B) + \sin^2 C \\ &= 1 + \cos C \cos(A - B) + 1 - \cos^2 C \\ &= 2 + \cos C [\cos(A - B) - \cos C] \\ &= 2 + \cos C [\cos(A - B) + \cos(A + B)] \\ &= 2 + 2 \cos C \cos A \cos B = \text{RHS} \end{aligned}$$

**Example : 23**

If  $A + B + C = \pi$ , Show that  $\cos A/2 + \cos B/2 + \cos C/2 = 4 \cos(\pi - A)/4 \cos(\pi - B)/4 \cos(\pi - C)/4$ .

**Solution**

$$\begin{aligned}
 \text{LHS} &= \cos \frac{A}{2} + \cos \frac{B}{2} + \cos \frac{C}{2} = 2 \cos \frac{A+B}{4} \cos \frac{A-B}{4} + \cos \frac{C}{2} \\
 &= 2 \cos \frac{\pi-C}{4} \cos \frac{A-B}{4} + \sin \frac{\pi-C}{2} \\
 &= 2 \cos \frac{\pi-C}{4} \cos \frac{A-B}{4} + 2 \sin \frac{\pi-C}{4} \cos \frac{\pi-C}{4} \\
 &= 2 \cos \frac{\pi-C}{4} \left[ \cos \frac{A-B}{4} + \sin \frac{\pi-C}{4} \right] \\
 &= 2 \cos \frac{\pi-C}{4} \left[ \cos \frac{A-B}{4} + \cos \left( \frac{\pi}{2} - \frac{\pi-C}{4} \right) \right] \\
 &= 2 \cos \frac{\pi-C}{4} \left[ 2 \cos \frac{\pi+A+C-B}{8} \cos \frac{A-B-\pi-C}{8} \right] \\
 &= 4 \cos \frac{\pi-C}{4} \left[ \cos \frac{A-C}{4} \cos \frac{B-C}{4} \right] \\
 &= 4 \cos \frac{\pi-C}{4} \cos \frac{\pi-B}{4} \cos \frac{\pi-A}{4} = \text{RHS}
 \end{aligned}$$

**Example : 24**

If  $x + y + z = xyz$ , then show that  $\frac{2x}{1-x^2} + \frac{2y}{1-y^2} + \frac{2z}{1-z^2} = \frac{2xyz}{(1-x^2)(1-y^2)(1-z^2)}$

**Solution**

Let  $x = \tan A$ ,  $y = \tan B$ ,  $z = \tan C$

$$\Rightarrow \tan(A+B+C) = \frac{\tan A + \tan B + \tan C - \tan A \tan B \tan C}{1 - \tan A \tan B - \tan B \tan C - \tan C \tan A} = \frac{x + y + z - xyz}{1 - xy - yz - zx}$$

$$\Rightarrow A + B + C = n\pi \quad (n \in \mathbb{I}) \quad (\tan \theta = 0 \Rightarrow \theta = n\pi)$$

$$\Rightarrow 2A + 2B + 2C = 2n\pi$$

$$\Rightarrow \tan(2A + 2B + 2C) = \tan 2n\pi = 0$$

$$\Rightarrow \frac{\tan 2A + \tan 2B + \tan 2C - \tan 2A \tan 2B \tan 2C}{1 - \tan 2A \tan 2B - \tan 2B \tan 2C - \tan 2C \tan 2A} = 0$$

$$\Rightarrow \tan 2A + \tan 2B + \tan 2C = \tan 2A \tan 2B \tan 2C$$

$$\Rightarrow \frac{2 \tan A}{1 - \tan^2 A} + \frac{2 \tan B}{1 - \tan^2 B} + \frac{2 \tan C}{1 - \tan^2 C} = \frac{2 \tan A}{1 - \tan^2 A} \cdot \frac{2 \tan B}{1 - \tan^2 B} \cdot \frac{2 \tan C}{1 - \tan^2 C}$$

$$\Rightarrow \frac{2x}{1-x^2} + \frac{2y}{1-y^2} + \frac{2z}{1-z^2} = \frac{2xyz}{(1-x^2)(1-y^2)(1-z^2)}$$

**Example : 25**

If  $xy + yz + zx = 1$ , the prove that  $\frac{x}{1+x^2} + \frac{y}{1+y^2} + \frac{z}{1+z^2} =$

**Solution**

Let  $x = \tan A/2, y = \tan B/2, z = \tan C/2$

$$\Rightarrow \tan A/2 \tan B/2 + \tan B/2 \tan C/2 + \tan C/2 \tan A/2 = 1$$

$$\Rightarrow \tan \left( \frac{A}{2} + \frac{B}{2} + \frac{C}{2} \right) \text{ is undefined}$$

$$\Rightarrow \frac{A}{2} + \frac{B}{2} + \frac{C}{2} = \frac{\pi}{2} \quad \Rightarrow \quad A + B + C = \pi$$

Using the relation :  $\sin \alpha + \sin \beta + \sin \gamma - \sin (\alpha + \beta + \gamma) = 4 \sin \left( \frac{\alpha + \beta}{2} \right) \sin \left( \frac{\beta + \gamma}{2} \right) \sin \left( \frac{\gamma + \alpha}{2} \right)$

Substitute  $\alpha = A, \beta = B, \gamma = C$

$$\Rightarrow \sin A + \sin B + \sin C = 4 \cos \frac{A}{2} \cos \frac{B}{2} \cos \frac{C}{2}$$

$$\Rightarrow \frac{2 \tan \frac{A}{2}}{1 + \tan^2 \frac{A}{2}} + \frac{2 \tan \frac{B}{2}}{1 + \tan^2 \frac{B}{2}} + \frac{2 \tan \frac{C}{2}}{1 + \tan^2 \frac{C}{2}} = 4 \cos \frac{A}{2} \cos \frac{B}{2} \cos \frac{C}{2}$$

$$\Rightarrow \frac{2x}{1+x^2} + \frac{2y}{1+y^2} + \frac{2z}{1+z^2} = 4 \cdot \frac{1}{\sqrt{1+y^2}} \cdot \frac{1}{\sqrt{1+z^2}}$$

$$\Rightarrow \frac{x}{1+x^2} + \frac{y}{1+y^2} + \frac{z}{1+z^2} = \frac{2xy}{\sqrt{(1+x^2)(1+y^2)(1+z^2)}}$$

**Example : 26**

If  $A + B + C = \pi$ , the show that :

$$\sin 3A \cos^3 (B - C) + \sin 3B \cos^3 (C - A) + \sin 3C \cos^3 (A - B) = \sin 3A \sin 3B \sin 3C$$

**Solution**

$$\text{LHS} = \sin 3A \cos^3(B - C) + \sin 3B \cos^3(C - A) + \sin 3C \cos^3(A - B)$$

$$= \sum \sin 3A \cos^3(B - C)$$

$$= \frac{1}{4} \sum \sin 3A [3 \cos(B - C) + \cos(3B - 3C)]$$

$$= \frac{3}{4} \sum \sin 3A \cos(B - C) + \frac{1}{4} \sum \sin 3A \cos(3B - 3C)$$

Substitute  $3A = 3\pi - (3B + 3C)$

$$= \frac{3}{4} \sum \sin(3B + 3C) \cos(B - C) + \frac{1}{4} \sum \sin(3B + 3C) \cos(3B - 3C)$$

$$= \frac{3}{8} \sum [\sin(4B + 2C) + \sin(2B + 4C)] + \frac{1}{8} \sum [\sin 6B + \sin 6C]$$

$$= \frac{3}{8} (0) + \frac{2}{8} (4 \sin 3C \sin 3B \sin 3A) = \sin 3A \sin 3B \sin 3C$$

**Example : 27**

Find the values of  $x$  lying between 0 and  $2\pi$  and satisfying the equation  $\sin x + \sin 3x = 0$ .

**Solution**

The given equation is  $\sin x + \sin 3x = 0$

$$2 \sin \frac{x+3x}{2} \cos \frac{x-3x}{2} = 0$$

$$\Rightarrow 2 \sin 2x \cos x = 0$$

$$\text{Hence } \sin 2x = 0 \quad \text{or} \quad \cos x = 0$$

$$\Rightarrow 2x = n\pi, n \in \mathbb{I} \quad \text{or} \quad x = (2n+1)\pi/2, n \in \mathbb{I}$$

$$\Rightarrow x = n\pi/2, n \in \mathbb{I} \quad \text{or} \quad x = (2n+1)\pi/2, n \in \mathbb{I}$$

$$(i) \quad n = 0 \Rightarrow x = 0, \pi/2$$

$$(ii) \quad n = 1 \Rightarrow x = \pi/2, 3\pi/2$$

$$(iii) \quad n = 2 \Rightarrow x = \pi, 5\pi/2$$

$$(iv) \quad n = 3 \Rightarrow x = 3\pi/2, 7\pi/2$$

Hence for  $0 < x < 2\pi$ , the solution is :

$$x = \pi/2, \pi, 3\pi/2$$

**Example : 28**

Find the values of  $\theta$  satisfying  $\sin \theta = \sin \alpha$

**Solution**

$$\sin \theta = \sin \alpha$$

$$\Rightarrow \sin \theta - \sin \alpha = 0$$

$$\Rightarrow 2 \cos \frac{\theta+\alpha}{2} \sin \frac{\theta-\alpha}{2} = 0$$

$$\Rightarrow \cos \frac{\theta+\alpha}{2} = 0 \quad \text{or} \quad \sin \frac{\theta-\alpha}{2} = 0$$

$$\Rightarrow \frac{\theta+\alpha}{2} = (2\ell+1)\frac{\pi}{2} \quad \text{or} \quad \frac{\theta-\alpha}{2} = n\pi \quad (\text{where } \ell, n \text{ are integers})$$

$$\theta = (2\ell+1)\pi - \alpha \quad \text{or} \quad \theta = 2n\pi + \alpha$$

$$\theta = (\text{odd no.})\pi - \alpha \quad \text{or} \quad \theta = (\text{even no.})\pi + \alpha$$

$$\theta = (\text{integer})\pi + (-1)^{\text{integer}}\alpha$$

$$\theta = n\pi + (-1)^n\alpha; n \in \mathbb{I}$$

**Example : 29**

Find the values of  $\theta$  satisfying  $\cos \theta = \cos \alpha$  in the interval  $0 \leq \theta \leq \pi$ .

**Solution**

$$\cos \theta = \cos \alpha$$

$$\Rightarrow \cos \theta - \cos \alpha = 0$$

$$\Rightarrow -2 \sin \frac{\theta+\alpha}{2} \sin \frac{\theta-\alpha}{2} = 0$$

$$\Rightarrow \sin \frac{\theta+\alpha}{2} = n\pi \quad \text{or} \quad \frac{\theta-\alpha}{2} = n\pi$$

$$\Rightarrow \theta = 2n\pi - \alpha \quad \text{or} \quad \theta = 2n\pi + \alpha$$

combining the two values :

$$\theta = 2n\pi \pm \alpha; \quad n \in \mathbb{I}$$

**Example : 30**

Find the values of  $\theta$  satisfying  $\tan \theta = \tan \alpha$

**Solution**

$$\tan \theta = \tan \alpha$$

$$\Rightarrow \frac{\sin \theta}{\cos \theta} = \frac{\sin \alpha}{\cos \alpha}$$

$$\Rightarrow \sin \theta \cos \alpha - \cos \theta \sin \alpha = 0$$

$$\Rightarrow \sin (\theta - \alpha) = 0$$

$$\Rightarrow \theta - \alpha = n\pi, n \in I$$

$$\Rightarrow \theta = n\pi + \alpha, n \in I$$

Note : The following results should be committed to memory before proceeding further.

$$(i) \quad \sin \theta = \sin \alpha \quad \Rightarrow \quad \theta = n\pi + (-1)^n \alpha, n \in I$$

$$(ii) \quad \cos \theta = \cos \alpha \quad \Rightarrow \quad \theta = 2n\pi \pm \alpha, n \in I$$

$$(iii) \quad \tan \theta = \tan \alpha \quad \Rightarrow \quad \theta = n\pi + \alpha, n \in I$$

Every trigonometric equation should be manipulated so that it reduces to any of the above results.

**Example : 31**

Solve the equation  $\cos x + \cos 2x + \cos 4x = 0$ , where  $0 \leq x \leq \pi$

**Solution**

$$\cos x + (\cos 2x + \cos 4x) = 0$$

$$\Rightarrow \cos x + 2 \cos 3x \cos x = 0$$

$$\Rightarrow \cos x (1 + 2 \cos 3x) = 0$$

$$\Rightarrow \cos x = 0 \quad \text{or} \quad 1 + 2 \cos 3x = 0$$

$$\Rightarrow \cos x = 0 \quad \text{or} \quad \cos 3x = -1/2 = \cos 2\pi/3$$

$$\Rightarrow x = (2n + 1) \pi/2 \quad \text{or} \quad 3x = 2n\pi \pm 2\pi/3$$

$$\Rightarrow x = (2n + 1) \pi/2, n \in I \quad \text{or} \quad x = 2n\pi/3 \pm 2\pi/9, n \in I$$

This is the general solution of the equation. To get particular solution satisfying  $0 \leq x \leq \pi$ , we will substitute integral values of  $n$ .

$$(i) \quad n = 0 \quad \Rightarrow \quad x = \pi/2, \pm 2\pi/9$$

$$(ii) \quad n = 1 \quad \Rightarrow \quad x = 3\pi/2, 8\pi/9, 4\pi/9$$

$$(iii) \quad n = 2 \quad \Rightarrow \quad x = 5\pi/2, 14\pi/9, 10\pi/9 \text{ (greater than } \pi)$$

$$(iv) \quad n = -1 \quad \Rightarrow \quad x = -\pi/2, -2\pi/3 \pm 2\pi/9 \text{ (less than } 0)$$

Hence the values for  $0 \leq x \leq \pi$  are

$$x = \pi/2, 2\pi/9, 4\pi/9, 8\pi/9$$

**Example : 32**

Solve the equation  $\sin x = \cos 4x$  for  $0 \leq x \leq \pi$

**Solution**

$$\sin x = \cos 4x$$

$$\Rightarrow \cos 4x = \cos (\pi/2 - x)$$

$$\Rightarrow 4x = 2n\pi \pm (\pi/2 - x)$$

$$\Rightarrow 4x = 2n\pi + \pi/2 - x \quad \text{or} \quad 4x = 2n\pi - \pi/2 + x$$

$$\Rightarrow x = 2n\pi/5 = \pi/10 \quad \text{or} \quad x = 2n\pi/3 - \pi/6$$

$$(i) \quad n = 0 \quad \Rightarrow \quad x = \pi/10, -\pi/6$$

$$(ii) \quad n = 1 \quad \Rightarrow \quad x = \pi/2$$

$$(iii) \quad n = 2 \quad \Rightarrow \quad x = 9\pi/10, 7\pi/6$$

$$(iv) \quad n = 3 \quad \Rightarrow \quad x = 13\pi/10, 11\pi/6 \text{ (greater than } \pi)$$

$$(v) \quad n = -1 \quad \Rightarrow \quad x = -3\pi/10, -5\pi/6 \text{ (less than } 0)$$

Hence the required solution for  $0 \leq x \leq \pi$  is :  $x = \pi/10, \pi/2, 9\pi/10$

**Example : 33**

Solve the equation  $\sqrt{3} \sin x + \cos x = 1$  in the interval  $0 \leq x \leq 2\pi$ .

**Solution**

For the equation of the type  $a \sin \theta + b \cos \theta = c$ ,

$$\text{write } a \sin \theta + b \cos \theta \quad \text{as} \quad \sqrt{a^2 + b^2} \cos (\theta - \alpha)$$

$$\sqrt{3} \sin x + \cos x = 1$$

$$\begin{aligned}
\Rightarrow & 2(\sqrt{3}/2 \sin x + 1/2 \cos x) = 1 \\
\Rightarrow & 2(\cos \pi/3 \cos x + \sin \pi/3 \sin x) = 1 \\
\Rightarrow & 2 \cos (x - \pi/3) = 1 \\
\Rightarrow & \cos (x - \pi/3) = \cos \pi/3 \\
\Rightarrow & x - \pi/3 = 2n\pi \pm \pi/3 \\
\Rightarrow & x = 2n\pi \pm \pi/3 + \pi/3 \\
\text{(i)} & n = 0 \Rightarrow x = 0, 2\pi/3 \\
\text{(ii)} & n = 1 \Rightarrow x = 2\pi + 2\pi/3, 2\pi \\
\text{(iii)} & n = 2 \Rightarrow x = 4\pi + 2\pi/3, 3\pi \text{ (greater than } 2\pi) \\
\text{(iv)} & n = -1 \Rightarrow x = -4\pi/3, -2\pi \text{ (less than } 0) \\
\end{aligned}$$

Hence the required values of x are 0, 2π/3, 2π

**Example : 34**

Solve  $\tan \theta + \tan 2\theta + \tan 3\theta = 0$  for general values of  $\theta$ .

**Solution**

Using  $\tan (A + B)$ ,  $\tan \theta + \tan 2\theta = \tan 3\theta (1 - \tan \theta \tan 2\theta)$

Hence the equation can be written as :

$$\tan 3\theta (1 - \tan \theta \tan 2\theta) + \tan 3\theta = 0$$

$$\tan 3\theta (2 - \tan \theta \tan 2\theta) = 0$$

$$\Rightarrow \tan 3\theta = 0 \quad \text{or} \quad \tan \theta \tan 2\theta = 2$$

$$\Rightarrow 3\theta = n\pi \quad \text{or} \quad 2 \tan^2 \theta = 2 (1 - \tan^2 \theta)$$

$$\Rightarrow \theta = n\pi/3, n \in I \quad \text{or} \quad \tan \theta = \pm 1/\sqrt{2}$$

$$\Rightarrow \theta = n\pi/3, n \in I \quad \text{or} \quad \theta = n\pi \pm \tan^{-1} 1/\sqrt{2}$$

**Example : 35**

Solve the equation  $\sin x + \cos x = \sin 2x - 1$

**Solution**

Let  $t = \sin x + \cos x$

$$\Rightarrow t^2 = 1 + 2 \sin x \cos x$$

$$\Rightarrow \sin 2x = t^2 - 1$$

Hence the given equation is :

$$t = (t^2 - 1) - 1$$

$$\Rightarrow t^2 - t - 2 = 0$$

Solving the equation,  $(t - 2)(t + 1) = 0$

$$\Rightarrow t = 2 \text{ or } t = -1$$

$$\Rightarrow \sin x + \cos x = 2 \quad \text{or} \quad \sin x + \cos x = -1$$

$$\Rightarrow \sqrt{2} \cos (x - \pi/4) = 2 \quad \text{or} \quad \sqrt{2} \cos (x - \pi/4) = -1$$

$$\Rightarrow \cos (x - \pi/4) = \sqrt{2} \quad \text{or} \quad \cos (x - \pi/4) = -1/\sqrt{2}$$

As  $-1 \leq \cos \theta \leq 1$ ,  $\cos (x - \pi/4) = \sqrt{2}$  is impossible.

$$\Rightarrow \cos (x - \pi/4) = -1/\sqrt{2} \text{ is the only possibility.}$$

$$\Rightarrow \cos (x - \pi/4) = \cos (\pi - \pi/4)$$

$$\Rightarrow x - \pi/4 = 2n\pi \pm 3\pi/4$$

$$\Rightarrow x = 2n\pi \pm 3\pi/4 + \pi/4 \text{ is the general solution}$$

**Example : 36**

Solve  $\sin^4 x + \cos^4 x = 7/2 \sin x \cos x$

**Solution**

$$\sin^4 x + \cos^4 x = 7/2 \sin x \cos x$$

$$\Rightarrow (\sin^2 x + \cos^2 x)^2 - 2 \sin^2 x \cos^2 x = 7/2 \sin x \cos x$$

$$\text{let } t = 2 \sin x \cos x = \sin 2x$$

$$\Rightarrow 1 - \frac{2t^2}{4} = \frac{7t}{4}$$

$$\Rightarrow 2t^2 + 7t - 4 = 0$$

$$\Rightarrow (2t - 1)(t + 4) = 0$$

$$\Rightarrow t = 1/2 \quad \text{or} \quad t = -4$$

$$\Rightarrow \sin 2x = 1/2 \quad \text{or} \quad \sin 2x = -4 \text{ (impossible)}$$

$$\begin{aligned} \Rightarrow \sin 2x &= \sin \pi/6 \\ \Rightarrow 2x &= n\pi + (-1)^n \pi/6, n \in I \\ \Rightarrow x &= n\pi/2 + (-1)^n \pi/12 \text{ is the general solution.} \end{aligned}$$

**Example : 37**

Put  $\cos 2x = 2 \cos^2 x - 1$

$$\begin{aligned} \Rightarrow 3 - 2 \cos x - 4 \sin x - (2 \cos^2 x - 1) + \sin 2x &= 0 \\ \Rightarrow (4 - 4 \sin x) - 2 \cos^2 x - 2 \cos x + \sin 2x &= 0 \\ \Rightarrow 4(1 - \sin x) - 2(1 - \sin^2 x) - 2 \cos x (1 - \sin x) &= 0 \\ \Rightarrow (1 - \sin x) (2 - 2 \sin x - 2 \cos x) &= 0 \\ \Rightarrow \sin x = 1 &\quad \text{or} \quad \sin x + \cos x = 1 \\ \Rightarrow \sin x = \sin \pi/2 &\quad \text{or} \quad \sqrt{2} \cos (x - \pi/4) = 1 \\ \Rightarrow x = n\pi + (-1)^n \pi/2 &\quad \text{or} \quad x - \pi/4 = 2n\pi \pm \pi/4 \\ \Rightarrow x = n\pi + (-1)^n \pi/2 &\quad \text{or} \quad x = 2n\pi \pm \pi/4 + \pi/4 \\ \Rightarrow x = n\pi + (-1)^n \pi/2 &\quad \text{or} \quad x = 2n\pi, 2n\pi + \pi/2 \end{aligned}$$

Combining the two, we get  $x = 2n\pi, 2n\pi + \pi/2$

**Example : 38**

Solve the inequality  $\sin x + \cos 2x > 1$  if  $0 \leq x \leq \pi/2$

**Solution**

Let  $\sin x = t$

$$\begin{aligned} \Rightarrow \cos 2x &= 1 - 2t^2 \\ \Rightarrow \text{the inequality is : } t + 1 - 2t^2 &> 1 \\ \Rightarrow t - 2t^2 &> 0 \\ \Rightarrow 2t^2 - t &< 0 \\ \Rightarrow t(2t - 1) &< 0 \\ \Rightarrow (t - 0) (t - 1/2) &< 0 \\ \Rightarrow 0 < t < 1/2 \\ \Rightarrow 0 < \sin x < 1/2 \\ \text{In } 0 \leq x \leq \pi/2, \text{ this means that } 0 < x < \pi/6 &\text{ is the solution} \end{aligned}$$

**Example : 39**

Find the principal and general solution of the equation :  $\sqrt{3}/2 \sin x - \cos x = \cos^2 x$

**Solution**

$$\begin{aligned} \sqrt{3}/2 \sin x - \cos x &= \cos^2 x \\ \text{squaring, } 3(1 - \cos^2 x) - 4 \cos^2 x (1 + \cos x)^2 &= 0 \\ \Rightarrow (1 + \cos x) [3 - 3 \cos x - 4 \cos^2 x (1 + \cos x)] &= 0 \\ \Rightarrow (1 + \cos x) [4 \cos^3 x + 4 \cos^2 x + 3 \cos x - 3] &= 0 \\ (1 + \cos x) [2 \cos x - 1] [2 \cos^2 x + 3 \cos x + 3] &= 0 \end{aligned}$$

The quadratic factor has no real roots

$$\begin{aligned} \Rightarrow \cos x = -1 &\quad \text{or} \quad \cos x = 1/2 \\ \Rightarrow x = (2n - 1) \pi &\quad \text{or} \quad x = 2n\pi \pm \pi/3 \end{aligned}$$

As we have squared the original equation, we must check whether these values satisfy the given equation. On checking, it is found that both solutions are accepted.

$$\Rightarrow x = (2n - 1) \pi, 2n\pi \pm \pi/3 \quad \text{where } n \in I$$

**Example : 40**

Solve for  $x$  ;  $\sec 4x - \sec 2x = 2$  ;  $-\pi \leq x \leq \pi$

**Solution**

$$\begin{aligned} \sec 4x - \sec 2x &= 2 \\ \Rightarrow \frac{1}{\cos 4x} - \frac{1}{\cos 2x} &= 2 \\ \Rightarrow \cos 2x - \cos 4x &= 2 \cos 2x \cos 4x \\ \Rightarrow \cos 2x - \cos 4x &= \cos 6x + \cos 2x \\ \Rightarrow \cos 6x + \cos 4x &= 0 \\ \Rightarrow 2 \cos 5x \cos x &= 0 \end{aligned}$$

$$\begin{aligned} \Rightarrow \cos 5x &= 0 & \text{or} & \cos x = 0 \\ \Rightarrow 5x &= n\pi + \pi/2 & \text{or} & x = n\pi + \pi/2 \\ \Rightarrow x &= n\pi/5 + \pi/10 & \text{or} & x = n\pi + \pi/2 \end{aligned}$$

Consider :  $x = n\pi/5 + \pi/10$  :

$$\begin{aligned} n = 0 & \Rightarrow x = \pi/10 \\ n = \pm 1 & \Rightarrow x = 3\pi/10, -\pi/10 \\ n = \pm 2 & \Rightarrow x = \pi/2, -3\pi/10 \\ n = \pm 3 & \Rightarrow x = 7\pi/10, -\pi/2 \\ n = \pm 4 & \Rightarrow x = 9\pi/10, -7\pi/10 \\ n = \pm 5 & \Rightarrow x = -9\pi/10 \end{aligned}$$

Consider :  $x = n\pi + \pi/2$

$$\begin{aligned} n = 0 & \Rightarrow x = \pi/2 \\ n = \pm 1 & \Rightarrow x = -\pi/2 \end{aligned}$$

These are the only values of  $x$  in  $[-\pi, \pi]$

### Example : 41

Solve the following equation for  $x$  if  $a$  is a constant.  $\sin 3a = 4 \sin a \sin (x + a) \sin (x - a)$

### Solution

$$\begin{aligned} \sin 3\alpha &= 4 \sin \alpha (\sin^2 x - \sin^2 \alpha) \\ \Rightarrow 3 \sin \alpha - 4 \sin^2 \alpha &= 4 \sin \alpha \sin^2 x - 4 \sin^3 \alpha \\ \Rightarrow 3 \sin \alpha - 4 \sin \alpha \sin^2 x &= 0 \\ \Rightarrow \sin \alpha (3 - 4 \sin^2 x) &= 0 \end{aligned}$$

If  $\sin \alpha = 0$ , then the equation is true for all real values of  $x$ .

If  $\sin \alpha \neq 0$ , then  $3 - 4 \sin^2 x = 0$

$$\begin{aligned} \Rightarrow \sin^2 x &= \sin^2 \pi/3 \\ \Rightarrow x &= n\pi \pm \pi/3 \end{aligned}$$

Note : The following results are very useful

1.  $\cos \theta = 0 \Rightarrow \theta = n\pi + \pi/2$
2.  $\sin \theta = 1 \Rightarrow \theta = 2n\pi + \pi/2$
3.  $\sin \theta = -1 \Rightarrow \theta = 2n\pi - \pi/2$
4.  $\cos \theta = 1 \Rightarrow \theta = 2n\pi$
5.  $\cos \theta = -1 \Rightarrow \theta = 2n\pi + \pi$
6.  $\sin^2 \theta = \sin^2 \alpha \Rightarrow \theta = n\pi \pm \alpha$
7.  $\cos^2 \theta = \cos^2 \alpha \Rightarrow \theta = n\pi \pm \alpha$
8.  $\tan^2 \theta = \tan^2 \alpha \Rightarrow \theta = n\pi \pm \alpha$

### Example : 42

If  $\tan (A - B) = 1$ ,  $\sec (A + B) = 2\sqrt{3}$ , calculate the smallest positive values and the most general values of  $A$  and  $B$ .

### Solution

Smallest positive values

Let  $A, B \in (0, 2\pi)$

$$\Rightarrow (A + B) > (A - B)$$

Now  $\tan (A - B) = 1$

$$\Rightarrow (A - B) = \pi/4, 5\pi/4$$

$\sec (A + B) = 2/\sqrt{3}$

$$\Rightarrow (A + B) = \pi/6, 11\pi/6$$

As  $(A + B) > (A - B)$ , these are two possibilities :

1.  $A - B = \pi/4$  &  $A + B = 11\pi/6$
2.  $A - B = 5\pi/4$  &  $A + B = 11\pi/6$

$$\text{From (i), we get : } A = \frac{25\pi}{24} \text{ and } B = \frac{19\pi}{24}$$

$$\text{From (ii), we get : } A = \frac{37\pi}{24} \text{ and } B = \frac{7\pi}{24}$$

General Values

$$\tan(A - B) = 1 \quad \Rightarrow \quad A - B = n\pi + \pi/4$$

$$\sec(A + B) = \frac{2}{\sqrt{3}} \quad \Rightarrow \quad A + B = n\pi + \pi/4$$

Taking  $A - B = n\pi + \frac{\pi}{4}$  and  $A + B = 2k\pi + \frac{\pi}{6}$  we get :

$$A = \frac{(2k+n)\pi}{2} + \frac{5\pi}{24} \quad \text{and} \quad B = \frac{(2k-n)\pi}{2} - \frac{\pi}{24}$$

Taking  $A - B = n\pi + \frac{\pi}{4}$  and  $A + B = 2k\pi - \frac{\pi}{6}$  we get :

$$A = \frac{(2k+n)\pi}{2} + \frac{\pi}{24} \quad \text{and} \quad B = \frac{(2k-n)\pi}{2} - \frac{5\pi}{24}$$

**Example : 43**

$$\text{Solve for } \theta : \tan\left(\frac{\pi}{2}\sin\theta\right) = \cot\left(\frac{\pi}{2}\cos\theta\right)$$

**Solution**

$$\tan\left(\frac{\pi}{2}\sin\theta\right) = \tan\left(\frac{\pi}{2} - \frac{\pi}{2}\cos\theta\right)$$

$$\Rightarrow \quad \frac{\pi}{2}\sin\theta = n\pi + \frac{\pi}{2} - \frac{\pi}{2}\cos\theta$$

$$\Rightarrow \quad \sin\theta + \cos\theta = 2n + 1$$

$$\Rightarrow \quad \cos\left(\theta - \frac{\pi}{4}\right) = \frac{2n+1}{\sqrt{2}}$$

As cosine lies between 1 and -1,  $n = 0, -1$  are the only

$$\text{Possible values of } n \text{ for } -1 \leq \frac{2n+1}{\sqrt{2}} \leq 1$$

$$\Rightarrow \quad \cos\left(\theta - \frac{\pi}{4}\right) = \pm \frac{1}{\sqrt{2}} \quad \Rightarrow \quad \theta - \frac{\pi}{4} = 2k\pi \pm \cos^{-1}\left(\pm \frac{1}{\sqrt{2}}\right)$$

$$\Rightarrow \quad \theta = 2k\pi + \frac{\pi}{4} \pm \cos^{-1}\left(\pm \frac{1}{\sqrt{2}}\right)$$

$$\Rightarrow \quad \theta = 2k\pi \pm \frac{\pi}{2}, 2k\pi, 2k\pi + \pi$$

For  $\theta = 2k\pi \pm \pi/2$ , the equation becomes undefined.

Hence the solution is :  $\theta = 2k\pi, 2k\pi + \pi$

$$\Rightarrow \quad \theta = m\pi, \text{ where } m \in \mathbb{I}$$

**Example : 44**

If  $\sin A = \sin B$  and  $\cos A = \cos B$ , find the value of A in terms of B.

**Solution**

$$\sin A - \sin B \quad \Rightarrow \quad 2 \cos \frac{A+B}{2} \sin \frac{A-B}{2} = 0$$

$$\cos A = \cos B \Rightarrow 2 \sin \frac{A+B}{2} \sin \frac{A-B}{2} = 0$$

$$\text{Both the equation will be satisfied if } \sin \frac{A-B}{2} = 0$$

$$\Rightarrow \frac{A-B}{2} = n\pi$$

$$\Rightarrow A = 2n\pi + B \quad \text{where } n \in \mathbb{I}$$

**Example : 45**

Evaluate : (i)  $\sin^{-1} \sin 4\pi/3$  (ii)  $\cos^{-1} \cos 5\pi/4$  (iii)  $\tan^{-1} \tan 2\pi/3$

**Solution**

$$\begin{aligned} \text{(i)} \quad 4\pi/3 \text{ does not lie in the principal value branch of } \sin^{-1}x. \text{ Hence } \sin^{-1} \sin 4\pi/3 &\neq 4\pi/3 \\ \sin^{-1} \sin 4\pi/3 &= \sin^{-1} \sin (\pi + \pi/3) \\ &= \sin^{-1} (-\sin \pi/3) \\ &= \sin^{-1} \sin \pi/3 = -\pi/3 \\ \text{(ii)} \quad \cos^{-1} \cos 5\pi/4 &= \cos^{-1} \cos (\pi + \pi/4) \\ &= \cos^{-1} (-\cos \pi/4) \\ &= \pi - \cos^{-1} \cos \pi/4 \\ &= \pi - \pi/4 \\ &= 3\pi/4 \\ \text{(iii)} \quad \tan^{-1} \tan 2\pi/3 &= \tan^{-1} \tan (\pi - \pi/3) \\ &= \tan^{-1} (-\tan \pi/3) \\ &= \tan^{-1} \tan \pi/3 \\ &= -\pi/3 \end{aligned}$$

**Example : 46**

Show that  $\tan^{-1} 1/3 + \tan^{-1} 1/2 = \pi/4$

**Solution**

$$\begin{aligned} \text{LHS} &= \tan^{-1} 1/3 + \tan^{-1} 1/2 \\ &= \tan^{-1} \left( \frac{\frac{1}{3} + \frac{1}{2}}{1 - \frac{1}{3} \cdot \frac{1}{2}} \right) \quad \left( \because \frac{1}{3} \cdot \frac{1}{2} < 1 \right) \\ &= \tan^{-1} \left( \frac{5/6}{5/6} \right) = \tan^{-1} 1 = \frac{\pi}{4} = \text{RHS} \end{aligned}$$

**Example : 47**

Show that :  $\cos^{-1} 9/10 + \operatorname{cosec}^{-1} \frac{\sqrt{41}}{4} = \frac{\pi}{4}$

**Solution**

$$\begin{aligned} \text{LHS} &= \cos^{-1} 9/10 + \operatorname{cosec}^{-1} \frac{\sqrt{41}}{4} \\ &= \tan^{-1} \frac{1}{9} + \sin^{-1} \frac{4}{\sqrt{41}} \\ &= \tan^{-1} \frac{1}{9} + \tan^{-1} \frac{4}{5} \quad \left( \text{using } \sin^{-1} x = \tan^{-1} \frac{x}{\sqrt{1-x^2}} \right) \end{aligned}$$

$$= \tan^{-1} \left( \frac{\frac{1}{9} + \frac{4}{5}}{1 - \frac{1}{9} \cdot \frac{4}{5}} \right) \quad \left( \because xy = \frac{1}{9} \times \frac{4}{5} < 1 \right)$$

$$= \tan^{-1} \left( \frac{41}{41} \right) = \tan^{-1} 1 = \frac{\pi}{4}$$

**Example : 48**

Show that :

(i)  $2 \tan^{-1} x = \tan^{-1} \frac{2x}{1-x^2}, -1 < x < 1$

(ii)  $2 \tan^{-1} x = \sin^{-1} \frac{2x}{1+x^2}, -1 < x < 1$

(iii)  $2 \tan^{-1} x = \cos^{-1} \frac{1-x^2}{1+x^2}, x > 0$

**Solution**

(i) Let  $x = \tan \theta$ ,  $\frac{\pi}{4} < \theta < \frac{\pi}{2}$  (using  $-1 < x < 1$ )

$$\text{RHS} = \tan^{-1} \frac{2 \tan \theta}{1 - \tan^2 \theta} = \tan^{-1} \tan 2\theta$$

$$= 2\theta = 2 \tan^{-1} x = \text{LHS} \quad [\because 2\theta \in (-\pi/2, \pi/2) \text{ lies in the principal value branch of } \tan^{-1}x]$$

(ii) Let  $x = \tan \theta \Rightarrow \frac{\pi}{4} < \theta < \frac{\pi}{2}$  (using  $-1 < x < 1$ )

$$\text{RHS} = \tan^{-1} \frac{2 \tan \theta}{1 + \tan^2 \theta} = \sin^{-1} \sin 2\theta$$

$$= 2\theta \quad [\because 2\theta \in (-\pi/2, \pi/2) \text{ lies in the principal value branch of } \sin^{-1}x]$$

$$= 2 \tan^{-1} x = \text{LHS}$$

(iii) Let  $x = \tan \theta$ ,  $0 < \theta < \pi/2$  (using  $x > 0$ )

$$\text{RHS} = \cos^{-1} \left( \frac{1 - \tan^2 \theta}{1 + \tan^2 \theta} \right)$$

$$= \cos^{-1} \cos 2\theta$$

$$= 2\theta \quad [\because 2\theta \in (0, \pi) \text{ lies in the principal value branch of } \cos^{-1}x]$$

$$= 2 \tan^{-1} x = \text{LHS}$$

**Example : 49**

Prove that

(i)  $\tan^{-1} \left( \frac{\cos \theta - \sin \theta}{\cos \theta + \sin \theta} \right) = \frac{\pi}{4} - \theta$

(ii)  $\tan^{-1} \left( \frac{\sqrt{1+x^2} - \sqrt{1-x^2}}{\sqrt{1+x^2} + \sqrt{1-x^2}} \right) = \frac{\pi}{4} - \frac{1}{2} \cos^{-1} x^2$

**Solution**

(i) LHS =  $\tan^{-1} \left( \frac{\cos \theta - \sin \theta}{\cos \theta + \sin \theta} \right) = \tan^{-1} \left( \frac{1 - \tan \theta}{1 + \tan \theta} \right)$  (dividing by  $\cos \theta$ )

$$= \tan^{-1} \tan \left( \frac{\pi}{4} - \theta \right) = \frac{\pi}{4} - \theta \quad [\text{for } \theta \in (-\pi/4, 3\pi/4)]$$

(ii) Put  $x^2 = \cos 2\theta$  in LHS

$$\text{LHS} = \tan^{-1} \left( \frac{\sqrt{1+\cos 2\theta} - \sqrt{1-\cos 2\theta}}{\sqrt{1+\cos 2\theta} + \sqrt{1-\cos 2\theta}} \right) = \tan^{-1} \left( \frac{\cos \theta - \sin \theta}{\cos \theta + \sin \theta} \right)$$

$$= \tan^{-1} \tan \left( \frac{\pi}{4} - \theta \right)$$

$$= \frac{\pi}{4} - \theta \quad [\text{for } \theta \in (-\pi/4, 3\pi/4)]$$

$$= \frac{\pi}{4} - \frac{1}{2} \cos^{-1} x^2 \quad [\text{using } x^2 = \cos 2\theta]$$

RHS

**Example : 50**

$$\begin{aligned} \text{Show that : } 2 \tan^{-1} \left[ \frac{\sqrt{a-b} \sqrt{1-\cos \theta}}{\sqrt{a+b} \sqrt{1+\cos \theta}} \right] &= \cos^{-1} \left[ \frac{1 - \left( \frac{a-b}{a+b} \right) \left( \frac{1-\cos \theta}{1+\cos \theta} \right)}{1 + \left( \frac{a-b}{a+b} \right) \left( \frac{1-\cos \theta}{1+\cos \theta} \right)} \right] \\ &= \cos^{-1} \left[ \frac{(a+b)(1+\cos \theta) - (a-b)(1-\cos \theta)}{(a+b)(1+\cos \theta) + (a-b)(1-\cos \theta)} \right] = \cos^{-1} \left[ \frac{a \cos \theta + b}{a + b \cos \theta} \right] = \text{RHS} \end{aligned}$$

**Example : 51**

$$\text{Show that : } 2 \tan^{-1} \left[ \tan \left( \frac{\pi}{4} - \frac{\alpha}{2} \right) \tan \left( \frac{\pi}{4} - \frac{\beta}{2} \right) \right] = \tan^{-1} \left[ \frac{\cos \alpha \cos \beta}{\sin \beta + \sin \alpha} \right]$$

**Solution**

$$\text{LHS} = 2 \tan^{-1} \left[ \frac{\sqrt{1-\sin \alpha} \sqrt{1-\sin \beta}}{\sqrt{1+\sin \alpha} \sqrt{1+\sin \beta}} \right] \quad \text{Using } \sqrt{\frac{1-\sin \theta}{1+\sin \theta}} = \tan \left( \frac{\pi}{4} - \frac{\theta}{2} \right)$$

$$= \tan^{-1} \left[ \frac{2 \sqrt{\left( \frac{1-\sin \alpha}{1+\sin \alpha} \right) \left( \frac{1-\sin \beta}{1+\sin \beta} \right)}}{1 - \left( \frac{1-\sin \alpha}{1+\sin \alpha} \right) \left( \frac{1-\sin \beta}{1+\sin \beta} \right)} \right]$$

$$= \tan^{-1} \left[ \frac{2 \sqrt{(1-\sin^2 \alpha)(1-\sin^2 \beta)}}{(1+\sin \alpha)(1+\sin \beta) - (1-\sin \alpha)(1-\sin \beta)} \right]$$

$$= \tan^{-1} \left[ \frac{2 \cos \alpha \cos \beta}{2 \sin \alpha + 2 \sin \beta} \right] = \text{RHS}$$

**Example : 52**

Show that :  $\sin \cot^{-1} \cos \tan^{-1} x = \sqrt{\frac{x^2+1}{x^2+2}}$

**Solution**

$$\text{Let } x = \tan \theta \Rightarrow \cos \theta = \frac{1}{\sqrt{1+x^2}}$$

$$\text{LHS} = \sin \cot^{-1} \cos \theta = \sin \cot^{-1} \frac{1}{\sqrt{1+x^2}}$$

$$\text{Let } \frac{1}{\sqrt{1+x^2}} = \cot \alpha \Rightarrow \sin \alpha = \frac{\sqrt{1+x^2}}{\sqrt{1+x^2+1}}$$

$$\Rightarrow \text{LHS} = \sin \cot^{-1} \cos \alpha = \sin \alpha = \sqrt{\frac{1+x^2}{2+x^2}} = \text{RHS}$$

**Example : 53**

If  $\sin^{-1} \frac{2a}{1+a^2} + \sin^{-1} \frac{2b}{1+b^2} = 2 \tan^{-1} x$ , then show that  $x = \frac{a+b}{1-ab}$

**Solution**

The given relation is :  $\sin^{-1} \frac{2a}{1+a^2} + \sin^{-1} \frac{2b}{1+b^2} = 2 \tan^{-1} x$

$$\Rightarrow 2 \tan^{-1} a + 2 \tan^{-1} b = 2 \tan^{-1} x$$

$$\Rightarrow \tan^{-1} \left( \frac{a+b}{1-ab} \right) = \tan^{-1} x$$

$$\Rightarrow x = \frac{a+b}{1-ab}$$

**Example : 54**

Solve for x :  $\tan^{-1} \frac{1}{2x+1} + \tan^{-1} \frac{1}{4x+1} = \tan^{-1} \frac{2}{x^2}$ .

**Solution**

Equating the tan of both sides  $\tan \left[ \tan^{-1} \frac{1}{2x+1} + \tan^{-1} \frac{1}{4x+1} \right] = \tan^{-1} \frac{2}{x^2}$

$$\Rightarrow \frac{\frac{1}{2x+1} + \frac{1}{4x+1}}{1 - \frac{1}{(2x+1)(4x+1)}} = \frac{2}{x^2}$$

$$\Rightarrow \frac{6x+2}{(2x+1)(4x+1)-1} = \frac{2}{x^2}$$

$$\Rightarrow (3x+1)x^2 = 8x^2 + 6x$$

$$\Rightarrow 3x^3 - 7x^2 - 6x = 0$$

$$\Rightarrow x = 0, 3, -2/3$$

$x = 0$  and  $-2/3$  are rejected because they don't satisfy the equation

Note that for  $x = 0$ , RHS is undefined

$$\Rightarrow \text{the only solution is } x = 3.$$

**Example : 55**

Show that  $\sum_{n=1}^{\infty} \tan^{-1} \frac{1}{1+n+n^2} = \frac{\pi}{4}$

**Solution**

$$\begin{aligned} \text{LHS} &= \sum_{n=1}^{\infty} \tan^{-1} \frac{1}{1+n+n^2} \\ &= \sum_{n=1}^{\infty} \tan^{-1} \frac{\overline{n+1}-n}{1+(n+1)n} \\ &= \sum_{n=1}^{\infty} [\tan^{-1}(n+1) - \tan^{-1}n] \\ &= (\tan^{-1}2 - \tan^{-1}1) + (\tan^{-1}3 - \tan^{-1}2) + \dots \\ &= -\tan^{-1}1 + \tan^{-1} \infty \\ &= -\frac{\pi}{4} + \frac{\pi}{2} = \frac{\pi}{4} = \text{RHS} \end{aligned}$$