Example : 1

Prove the following results :

(i) $r = (s - a)\tan\frac{A}{2} = (s - b)\tan\frac{B}{2} = (s - c)\tan\frac{C}{2}$

(ii) $r_1 = s\tan\frac{A}{2}$, $r_2 = s\tan\frac{B}{2}$, $r_3 = s\tan\frac{C}{2}$

(iii) $r = 4R\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$

Solution

(i) $r = \frac{\Delta}{s} = (s - a)\frac{\Delta}{s(s-a)}$

$\Rightarrow \quad r = (s - a)\tan\frac{A}{2} \qquad \left(\text{using } \cot\frac{A}{2} = \frac{s(s-a)}{\Delta}\right)$

other results follows by symmetry.

(ii) $r_1 = \frac{\Delta}{s-a} = \frac{s\Delta}{s(s-a)} = s\tan\frac{A}{2}$

Other results follow by symmetry.

(iii) $\sin\frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{bc}}$; $\sin\frac{B}{2} = \sqrt{\frac{(s-c)(s-a)}{ca}}$; $\sin\frac{C}{2} = \sqrt{\frac{(s-a)(s-b)}{ba}}$

multiply the three results to get :

$\Rightarrow \quad \sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2} = \frac{(s-a)(s-b)(s-c)}{abc}$

$\Rightarrow \quad \sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2} = \left(\frac{\Delta^2}{s}\right)\left(\frac{\sum 1}{4R\Delta}\right) + 4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2})$

$\Rightarrow \quad \sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2} = \left(\frac{\Delta}{s}\right)\left(\frac{1}{4R}\right)$

$\Rightarrow \quad r = \frac{\Delta}{s} = 4R\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$

Example : 2

Show that in a triangle ΔABC : a cot A b cot B + c cot C = 2 (R + r).

Solution

LHS = $\sum$ 2R sin A cot A = 2R cos A

$\Rightarrow$ LHS = 2R (cos A + cos B + cos C)

$\Rightarrow$ LHS = 2R

$\Rightarrow$ LHS = $2R + 8R\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$

$\Rightarrow$ LHS = 2R + 2r = RHS (using the result of last Ex.)

Example : 3

Show that : $\frac{r_1}{bc} + \frac{r_2}{ca} \frac{r_3}{ba} = \frac{1}{r} - \frac{1}{2R}$

Solution

$$LHS = \frac{\Delta}{abc}\left(\frac{a}{s-a}+\frac{b}{s-b}+\frac{c}{s-c}\right)$$

$$LHS = \frac{\Delta}{abc}\left(\frac{a}{s-a}+\frac{b}{s-b}+\frac{c}{s-c}\right) + \frac{1}{2R} - \frac{1}{2R}$$

$$LHS = \frac{\Delta}{abc}\left(\frac{a}{s-a}+\frac{b}{s-b}+\frac{c}{s-c}\right) + \frac{2\Delta}{abc} - \frac{1}{2R}$$

$$LHS = \frac{\Delta}{abc}\left(\frac{a}{s-a}+1+\frac{b}{s-b}+1+\frac{c}{s-c}\right) - \frac{1}{2R}$$

$$LHS = \frac{\Delta}{abc}\left(\frac{a}{s-a}+\frac{b}{s-b}+\frac{c}{s-c}\right) - \frac{1}{2R}$$

$$LHS = \frac{\Delta}{abc}\left(\frac{s(2s-a-b)}{(s-a)(s-b)}+\frac{c}{s-c}\right) - \frac{1}{2R}$$

$$LHS = \frac{\Delta}{ab}\left(\frac{s^2-sc+s^2-as-bs+ab}{(s-a)(s-b)(s-c)}\right) - \frac{1}{2R}$$

$$LHS = \frac{\Delta}{ab}\left(\frac{2s^2-s(2s)+ab}{(s-a)(s-b)(s-c)}\right) - \frac{1}{2R}$$

$$LHS = \frac{\Delta}{ab}\ \frac{\Delta}{(s-a)(s-b)(s-c)} - \frac{1}{2R}$$

$$LHS = \frac{\Delta s}{\Delta^2} - \frac{1}{2R} = \frac{1}{r} - \frac{1}{2R} = RHS$$

Example : 4

In a ΔABC, show that :

1. $c^2 = (a - b)^2 \cos^2 \frac{C}{2} + (a + b)^2 \sin^2 \frac{C}{2}$

2. $a \sin\left(\frac{A}{2}+B\right) = (b + c) \sin \frac{A}{2}$

3. $(b + c) \cos A + (c + a) \cos B + (a + b) \cos C = a + b + c$

Solution

1. $RHS = (a - b)^2\left(\frac{1+\cos C}{2}\right) = (a + b)^2\left(\frac{1-\cos C}{2}\right)$

2. $RHS = \frac{1}{2}\ [(a - b)^2 + (a + b)^2] + \frac{1}{2}\ \cos C\ [(a - b)^2 - (a + b)^2]$

$RHS\ a^2 + b^2 + \frac{1}{2}\ \cos C\ (-4ab) = c^2$ (using cosine rule)

Note : Try to prove the same identity using sine rule on RHS

2. LHS = a sin $\left(\frac{A}{2}+B\right)$ = 2R sin A sin $\left(\frac{A}{2}+B\right)$ (using sine rule)

LHS = 2R $\left(2\sin\frac{A}{2}\cos\frac{A}{2}\right)$ sin $\left(\frac{A}{2}+B\right)$

LHS = 2R sin $\frac{A}{2}$ $\left[2\cos\frac{A}{2}\sin\left(\frac{A}{2}+B\right)\right]$

LHS = 2R sin $\frac{A}{2}$ [sin (A + B) – sin (–B)]

LHS = 2R sin $\frac{A}{2}$ [sin C + sin B]

LHS = sin $\frac{A}{2}$ [2R sin C + 2R sin B]

LHS = sin $\frac{A}{2}$ (c + b) = RHS

Note : Try to prove the same identity using RHS

3. LHS = (b + c) cos A + (c + a) cos B + (a + b) cos C
LHS = [c cos B + b cos C] + a [a cos C + c cos A] + [b cos A + a cos B]
LHS = a + b + c = RHS

Example :

In a ΔABC, prove that $(b^2 - c^2)$ cot A + $(c^2 - a^2)$ cot B + $(a^2 - b^2)$ cot C = 0

Solution

Starting from LHS

$= \sum(b^2 - c^2)\ \cot A$

$= 4R^2 \sum(\sin^2 B - \sin^2 C)\ \cot A$ (using sine rule)

$= 4R^2 \sum \sin(B - C)\sin(B - C)\ \cot A$

$= 4R^2 \sum \sin A \sin(B - C)\frac{\cos A}{\sin A}$

$= -2R^2 \sum 2\cos(B + C)\sin(B - C)$ (using cos A = – cos (B + C)

$= -2R^2 \sum(\sin 2B - \sin 2C)$

$= -2R^2$ [(sin 2B – sin 2C) + (sin 2C – sin 2A) + (sin 2A – sin 2B)]

= 0 = RHS

Example : 6

In a ΔABC, show that : (a + b + c) $\left[\tan\frac{A}{2}+\tan\frac{B}{2}\right]$ = 2c cot $\frac{C}{2}$

Solution

Starting from LHS

= (a + b + c) $\left[\frac{(s-b)(s-c)}{\Delta}+\frac{(s-c)(s-a)}{\Delta}\right]$

= $\left(\frac{a+b+c}{\Delta}\right)$ (s – c) [s – b + s – a]

$$= \left(\frac{s-c}{\Delta}\right) (a + b + c)(c)$$

$$= \frac{(s-c)}{\Delta} = 2c \left[\frac{s(s-c)}{\Delta}\right] = 2c \cot \frac{C}{2} = \text{RHS}$$

Example : 7

In a ΔABC, prove that :

(i) $r_1 + r_2 + r_3 - r = 4R$

(ii) $rr_1 + rr_2 + rr_3 = ab + bc + ca - s^2$

Solution

(i) Starting from LHS

$$= \left(\frac{\Delta}{s-a} + \frac{\Delta}{s-b}\right) + \left(\frac{\Delta}{s-c} - \frac{\Delta}{s}\right)$$

$$= \Delta \frac{(2s - \overline{a+b})}{(s-a)(s-b)} + \frac{\Delta(s - \overline{s-c})}{s(s-c)}$$

$$= \frac{\Delta c}{(s-a)(s-b)} + \frac{\Delta c}{s(s-c)}$$

$$= \frac{\Delta c}{s(s-a)(s-b)(s-c)} \; [ss - c) +) \; s - a) \; (s - b)]$$

$$= \frac{c}{\Delta} [2s^2 - 2s^2 + ab] = \frac{abc}{\Delta} = 4\left(\frac{abc}{4\Delta}\right) = 4R$$

(ii) Starting from LHS

$$= \frac{\Delta^2}{s} \left[\frac{1}{s-a} + \frac{1}{s-b} + \frac{1}{s-c}\right]$$

$$= \frac{\Delta^2}{s} \left[\frac{\sum (s-b)(s-c)}{(s-a)(s-b)(s-c)}\right]$$

$= 3s^2 - 2s\,(a + b + c) + bc + ca + ab$

$= 3s^2 - 4s^2 + bc + ca + ab$

$= ab + bc + ca - s^2 = \text{RHS}$

Example : 8

In a ΔABC, show that : $\dfrac{(a+b+c)^2}{a^2+b^2+c^2} = \dfrac{\cot\frac{A}{2} + \cot\frac{B}{2} + \cot\frac{C}{2}}{\cot A + \cot B + \cot C}$

Solution

Starting from RHS

$$= \frac{\frac{s(s-a)}{\Delta} + \frac{s(s-b)}{\Delta} + \frac{s(s-c)}{\Delta}}{\frac{b^2+c^2-a^2}{4\Delta} + \frac{c^2+a^2-b^2}{4\Delta} + \frac{a^2+b^2-c^2}{4\Delta}} = \frac{4s[s-a+s-b+s-c]}{b^2+c^2+a^2}$$

$$= \frac{4s(3s-2s)}{a^2+b^2+c^2} = \frac{4s^2}{a^2+b^2+c^2} = \frac{(a+b+c)^2}{a^2+b^2+c^2} \; \text{LHS}$$

Example : 9

If a^2 , b^2 , c^2 in a ΔABC are in A.P. Prove that cot A, cot B and cot C are also in A.P.

Solution

cot A, cot B and cot C are in A.P. if :

cot A – cot B = cot B – cot C

$$\Rightarrow \quad \frac{\cos A}{\sin A} - \frac{\cos B}{\sin B} = \frac{\cos B}{\sin B} - \frac{\cos C}{\sin C}$$

$$\Rightarrow \quad \frac{\sin(B-A)}{\sin A \sin B} = \frac{\sin(C-B)}{\sin B \sin C}$$

$\Rightarrow$ sin (B – A) sin C = sin (C – B) sin A

$\Rightarrow$ sin (B – A) sin (B + A) = sin (C – B) sin (C + B)

$\Rightarrow$ $\sin^2 B - \sin^2 A = \sin^2 C - \sin^2 B$

$$\Rightarrow \quad \frac{b^2}{4R^2} - \frac{a^2}{4R^2} = \frac{c^2}{4R^2} - \frac{b^2}{4R^2} \qquad \text{(using sine rule)}$$

$\Rightarrow$ $b^2 - a^2 = c^2 - b^2$ $\Rightarrow$ $ab^2 = a^2 + c^2$

$\Rightarrow$ a^2 , b^2 , c^2 are in A.P.

$\Rightarrow$ cot A, cot B and cot C are also in A.P.

Example : 10

If x, y, z are respectively the perpendiculars from circumcentre to the sides of the triangle ABC prove that

$$\frac{a}{x} + \frac{b}{y} + \frac{c}{z} = \frac{abc}{4xyz}$$

Solution

We known that : x = R cos A, y = R cos B, z = R cos C

Consider LHS :

$$= \frac{a}{R\cos A} + \frac{b}{R\cos B} + \frac{c}{R\cos C}$$

$$= \frac{2R\sin A}{R\cos A} + \frac{2R\sin B}{R\cos B} = \frac{2R\cos C}{R\cos C}$$

= (tan A + tan B + tan C)

= (tan A tan B tan C) $\qquad (\because A + B + C = \pi)$

$$= 2\left[\frac{\sin A}{\cos A}\frac{\sin B}{\cos B}\frac{\sin C}{\cos C}\right]$$

$$= \frac{2}{8R^3}\left[\frac{abc}{\cos A\cos B\cos C}\right] \qquad \text{(using sine rule)}$$

$$= \frac{1}{4}\left[\frac{abc}{(R\cos A)(R\cos B)(R\cos C)}\right] = \frac{1}{4}\,\frac{abc}{xyz} = \text{RHS}$$

Example : 11

I is the incentre of ΔABC and P_1 , P_2 , P_3 are respectively the radii of the circumcircle of DIBC, ΔICA and IAB, prove that : $P_1 P_2 P_3 = 2R^2 r$.

Solution

$$\angle BIC = \pi - \frac{1}{2}(B + C) = \pi - \frac{1}{2}(\pi - A) = \frac{\pi}{2} + \frac{A}{2}$$

Circumradius of ΔABC is :

$$P_1 = \frac{BC}{2\sin\angle BIC} = \frac{BC}{2\sin\left(\frac{\pi}{2}+\frac{A}{2}\right)} = \frac{a}{2\cos\frac{A}{2}}$$

Similarly we can show that : $P_2 = \dfrac{b}{2\cos\frac{B}{2}}$ and $P_3 = \dfrac{c}{2\cos\frac{C}{2}}$

$$\Rightarrow \quad P_1P_2P_3 = \frac{abc}{8\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}} = \frac{8R^2\sin A\sin B\sin C}{8\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}}$$

$$= \frac{8R^3\sin\frac{A}{2}\cos\frac{A}{2}\sin\frac{B}{2}\cos\frac{B}{2}\sin\frac{C}{2}\cos\frac{C}{2}}{\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}} = 8R^3\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$$

$= 2R^2 r = \text{RHS}$

Example : 12

(Ptolemy Theorem) If ABCD is cyclic quadrilateral, show that AC . BD = AB . CD + BC . AD

Solution

Let AB = a, BC = b, CD = c, DA = d

using cosine rule in ΔABC and ΔADC, we get :

$AC^2 = a^2 + b^2 - 2ab\cos B$

$AC^2 = c^2 + d^2 - 2cd\cos D$

and $B + D = \pi$

$\Rightarrow \quad \cos B + \cos D = 0$

$\Rightarrow \quad AC^2(cd + ab) = (a^2 + b^2)cd + (c^2 + d^2)ab$

$$\Rightarrow \quad AC^2 = \frac{(a^2cd + c^2ab) + (b^2cd + d^2ab)}{cd + ab}$$

Similarly by taking another diagonal BD, we can show that :

$$BD^2 = \frac{(ba + cd)(bd + ca)}{da + bc}$$

Multiplying the two equations

$\Rightarrow \quad (AD\ .\ BD)^2 = (ac + bd)^2$

$\Rightarrow \quad AC\ .\ BD = ac + bd$

$\Rightarrow \quad AC\ .\ BD = AB\ .\ CD + BC\ .\ AD$

Example : 13

Show that : $\left[\cot\frac{A}{2} + \cot\frac{B}{2}\right]\left[a\sin^2\frac{B}{2} + b\sin^2\frac{A}{2}\right] = c\cot\frac{C}{2}$

Solution

Taking LHS :

$$= \left[\frac{s(s-a)}{\Delta} + \frac{s(s-b)}{\Delta}\right]\left[\frac{a(s-c)(s-a)}{ca} + \frac{b(s-b)(s-c)}{bc}\right]$$

$$= \frac{s}{\Delta}[2s - a - b]\left(\frac{s-c}{c}\right)(2s - a - b)$$

$$= \frac{s(s-c)}{\Delta c}c^2 = c\,\frac{s(s-c)}{\Delta} = c\cot\frac{C}{2} = \text{RHS}$$

Example : 14

In a ΔABC, show that $a^3 \cos(B - C) + b^3 \cos(C - A) + c^3 \cos(A - B) = 3abc$

Solution

Given expression $= \sum a^3 \cos(B - C)$

$= \sum a^2 (2R \sin A) \cos(B - C)$

$= R \sum a^2 (2R \sin \overline{B + C} \cos \overline{B - C})$

$= R \sum a^2 (\sin 2B + \sin 2C)$

$= 2R \sum a^2 (\sin B + \sin B + \sin C \cos C) = \sum a^2 (b \cos B + c \cos C)$

$= a^2 (\underline{b \cos B} + \overline{c \cos C}) + b^2 (c \cos C + \underline{a \cos A}) + c^2 (\overline{a \cos A} + b \cos B)$

$= ab(a \cos B + b \cos A) + ac(a \cos C + a \cos A) + bc(b \cos C + c \cos B)$

$= abc + acb + bca$ (using projection formula)

$= 3abc = \text{RHS}$

Example : 15

If the sides a, b, c of a ΔABC are in A.P., then prove that $\cot \frac{A}{2}$, $\cot \frac{B}{2}$ and $\cot \frac{C}{2}$ are also in A.P.

Solution

a, b, c are in A.P. $\Rightarrow$ $a - b = b - c$

$\Rightarrow \quad \sin A - \sin B = \sin B - \sin C$

$\Rightarrow \quad 2 \cos \frac{A + B}{2} \sin \frac{A - B}{2} = 2 \cos \frac{B + C}{2} \sin \frac{B - C}{2}$

$\Rightarrow \quad \sin \frac{C}{2} \sin \frac{A - B}{2} = \sin \frac{A}{2} \sin \frac{B - C}{2}$

$$\Rightarrow \quad \frac{\sin\left(\frac{A}{2} - \frac{B}{2}\right)}{\sin\frac{A}{2}\sin\frac{B}{2}} = \frac{\sin\left(\frac{B}{2} - \frac{C}{2}\right)}{\sin\frac{B}{2}\sin\frac{C}{2}}$$

$\Rightarrow \quad \cot \frac{B}{2} - \cot \frac{A}{2} = \cot \frac{C}{2} - \cot \frac{B}{2}$

$\Rightarrow \quad \cot \frac{A}{2}, \cot \frac{B}{2}, \cot \frac{C}{2}$ are in A.P.

Example : 16

In a ΔABC, prove that $A = B$ if : $a \tan A + b \tan B = (a + b) \tan \left(\frac{A + B}{2}\right)$

Solution

Rearranging the terms of the given expression as follows :

$\Rightarrow \quad a \tan A - a \tan \frac{A + B}{2} - b \tan \frac{A + B}{2} - b \tan B$

$$\Rightarrow \quad \frac{a \sin\left(A - \frac{A + B}{2}\right)}{\cos A \cos \frac{A + B}{2}} = \frac{b \sin\left(\frac{A + B}{2} - B\right)}{\cos \frac{A + B}{2} \cos B}$$

$\Rightarrow \quad \dfrac{2R\sin A \sin\left(\dfrac{A-B}{2}\right)}{\cos A} = \dfrac{2R\sin B \sin\left(\dfrac{A-B}{2}\right)}{\cos B}$

$\Rightarrow \quad \sin\left(\dfrac{A-B}{2}\right)$ [tan A – tan B] = 0

$\Rightarrow \quad \sin\left(\dfrac{A-B}{2}\right)$ = 0 or tan A – tan B = 0

$\Rightarrow \quad A = B$

Example : 17

If the sides of a triangle are in A.P. and the greatest angle exceeds the smallest angle by α, show that the sides are in the ratio 1 – x : 1 : 1 + x; where $x = \sqrt{\dfrac{1-\cos\alpha}{7-\cos\alpha}}$

Solution

Let A > B > C

$\Rightarrow \quad A - C = \alpha \quad$ and $\quad ab = a + c$

We will first find the values of sin B/2 and cos B/2

2b = a + c

$\Rightarrow \quad 2\sin B = \sin A + \sin C$

$\Rightarrow \quad 4\sin\dfrac{B}{2}\cos\dfrac{B}{2} = 2\sin\dfrac{A+C}{2}\cos\dfrac{A-C}{2}$

$\Rightarrow \quad 4\sin\dfrac{B}{2}\cos\dfrac{B}{2} = 2\cos\dfrac{B}{2}\cos\dfrac{\alpha}{2}$

$\Rightarrow \quad \sin\dfrac{B}{2} = \dfrac{1}{2}\cos\dfrac{\alpha}{2} \quad \Rightarrow \quad \sin\dfrac{B}{2} = \sqrt{\dfrac{1+\cos\alpha}{2\sqrt{2}}}$

$\Rightarrow \quad \cos\dfrac{B}{2} = \sqrt{1-\sin^2\dfrac{B}{2}} = \dfrac{\sqrt{7-\cos\alpha}}{2\sqrt{2}}$ ....(i)

Consider

$\dfrac{a}{c} = \dfrac{\sin A}{\sin C}$ (using sine rule)

$\Rightarrow \quad \dfrac{a+c}{a-c} = \dfrac{\sin A + \sin C}{\sin A - \sin C}$

$\Rightarrow \quad \dfrac{a+c}{a-c} = \dfrac{2\sin B}{2\cos\dfrac{A+C}{2}\sin\dfrac{A-c}{2}}$

$\Rightarrow \quad \dfrac{a+c}{a-c} = \dfrac{2\left(2\sin\dfrac{B}{2}\cos\dfrac{B}{2}\right)}{2\sin\dfrac{B}{2}\sin\dfrac{\alpha}{2}}$

$\Rightarrow \quad \dfrac{a+c}{a-c} = 2\,\dfrac{\cos B/2}{\sin\alpha/2}$

$$\Rightarrow \quad \frac{a+c}{a-c} = \frac{2\left(\frac{\sqrt{7-\cos\alpha}}{2\sqrt{2}}\right)}{\sin\alpha/2} \qquad \text{(using (i))}$$

$$\Rightarrow \quad \frac{a+c}{a-c} = \frac{\sqrt{7-\cos\alpha}}{\sqrt{1-\cos\alpha}}$$

$$\Rightarrow \quad \frac{a+c}{a-c} = \frac{1}{x}$$

$$\Rightarrow \quad \frac{a}{c} = \frac{1+x}{1-x}$$

$$\Rightarrow \quad \frac{a}{1+x} = \frac{c}{1-x}$$

$$\Rightarrow \quad \frac{a}{1+x} = \frac{c}{1-x} = \frac{a+c}{2}$$

$$\Rightarrow \quad \frac{a}{1+x} = \frac{c}{1-x} = \frac{2b}{2}$$

$$\Rightarrow \quad \frac{a}{1+x} = \frac{b}{1} = \frac{c}{1-x}$$

Example : 18

Δ is the mid point of BC in a ΔABC. If AD is perpendicular to AC, show that : cos A cos C = $\frac{2(c^2-a^2)}{3ac}$

Solution

The value of cos C can be found by cosine rule in ΔABC or ΔADC

From ΔABC : cos C = $\frac{a^2+b^2-c^2}{2ab}$

From ΔADC : cos C = $\frac{a}{a/2}$

$$\Rightarrow \quad \frac{2b}{a} = \frac{a^2+b^2-c^2}{2ab}$$

$$\Rightarrow \quad b^2 = \frac{a^2-c^2}{3} \qquad \text{(i)}$$

$$= \text{LHS} = \cos A \cos C = \left(\frac{b^2+c^2-a^2}{2bc}\right)\left(\frac{b}{a/2}\right)$$

$$= \frac{b^2+c^2-a^2}{ac} = \frac{\frac{a^2-c^2}{3}+c^2-a^2}{ac} \qquad \text{(using (i))}$$

$$= \frac{2(c^2-a^2)}{3ac} = \text{RHS}$$

Example : 19

Let O be a point inside a $\angle$ABC such that $\angle OAB = \angle OBC = \angle OCA = \omega$, Show that

(i) $\cos\omega = \cot A + \cot B + \cot C$

(ii) $\text{cosec}^2\omega = \text{cosec}^2A + \text{cosec}^2B + \text{cosec}^2C$

Solution

Apply the sine rule in ΔOBC

$$\Rightarrow \quad \frac{OB}{a} = \frac{\sin(c-\omega)}{\sin(\pi-\omega+C-\omega)}$$

$$\Rightarrow \quad \frac{OB}{a} = \frac{\sin(C-\omega)}{\sin C} \qquad ....(i)$$

Applying sine rule in ΔOAB and proceeding similarly :

$$\Rightarrow \quad \frac{OB}{c} = \frac{\sin\omega}{\sin B}$$

Divide (i) by (ii) to get :

$$\frac{c}{a} = \frac{\sin(C-\omega)\sin B}{\sin\omega\sin C} \text{ (sine rule in } \Delta ABC)$$

$$\Rightarrow \quad \frac{\sin C}{\sin A\sin B} = \frac{\sin(C-\omega)}{\sin\omega\sin C}$$

$$\Rightarrow \quad \frac{\sin(A+B)}{\sin A\sin B} = \frac{\sin(C-\omega)}{\sin\omega\sin C}$$

$\Rightarrow$ $\cot B + \cot A = \cot\omega - \cot C$

$\Rightarrow$ $\cot\omega = \cot A + \cot B + \cot C$

(ii) Squaring the above result :

$\cot^2\omega = (\cot A + \cot B + \cot C)^2$

$\Rightarrow$ $\text{cosec}^2\omega - 1 = \sum \cot^2 A + 2 \quad \cot A \cot B$ $\sum \frac{+c^2-a^2}{2bc}$

$\Rightarrow$ $\text{cosec}^2\omega - 1 = \quad (\text{cosec}^2A - 1) + 2 \quad (\because \text{ in a } \Delta \quad \cot A \cot B = 1)$

$\Rightarrow$ $\text{cosec}^2\omega = \quad \text{cosec}^2A - 3 + 2$

$\Rightarrow$ $\text{cosec}^2\omega = \text{cosec}^2A + \text{cosec}^2B + \text{cosec}^2C$

Example : 20

For a triangle ABC, it is given that : cos A + cos B + cos C = 3/2. Prove that the triangle is equilateral.

Solution

Consider cos A + cos B + cos C = 3/2

$$\Rightarrow \quad + \frac{c^2+a^2+b^2}{2ca} + \frac{a^2+b^2+c^2}{2ab} = \frac{3}{2}$$

$\Rightarrow$ $2(b^2 + c^2 - a^2) + b(c^2 + a^2 - b^2) + c(a^2 + b^2 + c^2) = 3abc$

$\Rightarrow$ $a(b^2 + c^2) + b(c^2 + a^2) + c(a^2 + b^2) = a^3 + b^3 + c^3 + 3abc$

$\Rightarrow$ $a(b^2 + c^2 - 2bc) + b(c^2 + a^2 - 2ac) + c(a^2 + b^2 - 2ab) = a^3 + b^3 - 3abc$

$\Rightarrow$ $a(b-c)^2 + b(c-a)^2 + c(a-b)^2 - 1/2\,(a+b+c)\,[(b-c)^2 + (c-a)^2 + (a-b)^2] = 0$

$\Rightarrow$ $(b-c)^2(b+c-a) + (c-a)^2(c+a-b) + (a-b)^2(a+b-c) = 0$

$\Rightarrow$ $\because$ sum of two sides > third side

$\Rightarrow$ All terms in LHS are non-negative

$\Rightarrow$ each term = 0

$\Rightarrow$ $b - c = c - a = a - b = 0$

$\Rightarrow$ $a = b = c$

$\Rightarrow$ ΔABC is a equilateral.

Example : 21

If a = 100, c = 100 √2 and A = 30º, solve the triangle.

Solution

$a^2 = b^2 + c^2 - 2bc \cos A$

$b^2 - 2b\,(100\sqrt{2}) \cos 30^\circ + (100\sqrt{2})^2 - 100^2 = 0$

$b^2 - 100\sqrt{6}\,b + 10000 = 0$

$$b = \frac{100\sqrt{6} \pm 100\sqrt{2}}{2} = 50\sqrt{2}\left(\sqrt{3} \pm 1\right)$$

$b_1 = 50\sqrt{2}\,(\sqrt{3} - 1)$, $b_2 = 50\sqrt{2}\,(\sqrt{3} + 1)$

$$\sin C = \frac{c \sin A}{a} = \frac{100\sqrt{2} \sin 30^\circ}{100} = \frac{1}{\sqrt{2}}$$

$C_1 = 135^\circ$ and $C_2 = 45^\circ$

$B_1 = 180 - (135^\circ + 30^\circ) = 15^\circ$

$B_2 = 180 - (45^\circ + 30^\circ) = 105^\circ$

Example : 22

In the ambiguous case, if the remaining angles of the triangle formed with a, b and A be B1 , C_1 and B_2 , C_2,

then prove that : $\frac{\sin C_2}{\sin B_1} + \frac{\sin C_2}{\sin_2} = 2 \cos A$.

Solution

$$\sin B_1 - \sin B_2 = \frac{b \sin A}{a} \qquad \text{(using sine rule)}$$

$$\sin C_1 = \frac{c_1 \sin A}{a} \text{ and } \sin C_2 = \frac{c_2 \sin A}{a}$$

$$\Rightarrow \quad \text{LHS} = \frac{\frac{c_1 \sin A}{a}}{\frac{b \sin A}{a}} + \frac{\frac{c_2 \sin A}{a}}{\frac{b \sin A}{a}}$$

$$\Rightarrow \quad \text{LHS} = \frac{c_1 + c_2}{b} = \frac{2b \cos A}{b} = 2 \cos A$$

Example : 23

In a ΔABC; a, c, A are given and $b_1 = 2b_2$, where b_1 and b_2 are two values of the third side : then prove that:

$3a = c\sqrt{1 + 8\sin^2 A}$

Solution

$a^2 = b^2 + c^2 - 2bc \cos A$

consider this equation as a quadratic in b.

$\Rightarrow \quad b^2 - (2c \cos A)\, b + c^2 - a^2 = 0$

$\Rightarrow \quad b_1 + b_2 = 2c \cos A$

and $\quad b_1 - b_2 = c^2 - a^2$

and $\quad b_1 = 2b_2$

$\Rightarrow \quad 3b_1 = 2c \cos A$ and $2b_1^2 = c^2 - a^2$

$$\Rightarrow \quad 2\left(\frac{2c \cos A}{3}\right)^2 \; c^2 - a^2$$

$\Rightarrow \quad 8c^2 \cos^2 A = 9c^2 - 9a^2$

$\Rightarrow \quad 8c^2 (1 - \sin^2 A) = 9c^2 - 9a^2$

$\Rightarrow \quad 9a^2 = c^2 + 8c^2 \sin^2 A$

$\Rightarrow \quad 3a = c\sqrt{1 + 8\sin^2 A}$

Example : 24

A man observes, that when he moves up a distance c meters on a slope, the angle of depression of a point on the horizontal plane from the base of the slope is 30º; and when he moves up further a distance c meters the angle of depression of that point is 45º. Obtain the angle of elevation of the slope with the horizontal.

Solution

Let the point A be observed from Q and R

$\Rightarrow$ PQ = QR = c

Apply m – n theorem in ΔAPR. Q divides PR in ratio c : c

$\Rightarrow$ $(c + c) \cot (\theta - 30^\circ) = c \cot 15^\circ - c \cot 30^\circ$

$\Rightarrow$ $2 \cot (\theta - 30^\circ) = 2 + \sqrt{3} - \sqrt{3}$

$\Rightarrow$ $2 \cot (\theta - 30^\circ) = 2$

$\Rightarrow$ $\cot (\theta - 30^\circ) = 1$

$\Rightarrow$ $\theta - 30^\circ = 45^\circ$ $\Rightarrow$ $\theta = 75^\circ$

Example : 25

A vertical pole (more than 100 ft high) consists of two portions, the lower being one third of the whole. If the upper portion subtends an angle $\tan^{-1}(1/2)$ at a point in the horizontal plane through the foot of the pole and at a distance of 40ft from it, find the height of the pole.

Solution

Let PQ be the tower and R be the point dividing PQ in 1 : 2

Angle subtended at A = $\alpha = \tan^{-1}1/2$

$\Rightarrow$ $\alpha = \tan^{-1}\dfrac{PQ}{AP} - \tan^{-1}\dfrac{PR}{AP}$

$\Rightarrow$ $\tan^{-1}\dfrac{1}{2} = \tan^{-1}\dfrac{h}{40} - \tan^{-1}\dfrac{h/3}{40}$

$\Rightarrow$ $\dfrac{1}{2} = \dfrac{\dfrac{h}{40} - \dfrac{h}{120}}{1 + \dfrac{h^2}{4800}}$

$\Rightarrow$ $1 + \dfrac{h^2}{4800} = \dfrac{h}{20} - \dfrac{h}{60}$

$\Rightarrow$ $h^2 - 160h + 4800 = 0$

$\Rightarrow$ h = 40, 120

$\Rightarrow$ h = 120 ft. (as h > 100ft)

Example : 26

A 2 metre long object is fired vertically upwards from the mid-point of two locations A and B, 8 metres apart. The speed of the object after t seconds is given by ds/dt = 2t + 1 m/s. Let α and β be the angles subtended by the object at A and B respectively after one and two seconds. Find the value of $\cos(\alpha - \beta)$.

Solution

At t = 1 s :

$$OP = s = \int_0^1 (2t + 1)\, dt = 2m$$

$\Rightarrow$ $\alpha = \tan^{-1}\left(\dfrac{OP + PQ}{OA}\right) - \tan^{-1}\left(\dfrac{OP}{OA}\right)$

$\Rightarrow$ $\alpha = \tan^{-1}\left(\dfrac{2+2}{4}\right) - \tan^{-1}\left(\dfrac{2}{4}\right)$

$\Rightarrow \quad \tan\alpha = \frac{1}{3}$

At t = 2 s :

$$OP = \int_0^2 (2t+1)\, dt = 6m.$$

$$\Rightarrow \quad \beta = \tan^{-1}\left(\frac{6+2}{4}\right) - \tan^{-1}\left(\frac{6}{4}\right)$$

$$\Rightarrow \quad \tan\beta = \frac{2-3/2}{1+2.3/2}$$

$$\Rightarrow \quad \tan\beta = \frac{1}{8}$$

$$\Rightarrow \quad \tan\alpha = \frac{1}{3} \text{ and } \tan\beta = \frac{1}{8}$$

$$\Rightarrow \quad \sin\alpha = \frac{1}{\sqrt{10}}, \cos\alpha = \frac{3}{\sqrt{10}}$$

$$\Rightarrow \quad \sin\beta = \frac{1}{\sqrt{65}}, \cos\beta = \frac{8}{\sqrt{65}}$$

$$\Rightarrow \quad \cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta = \frac{3}{\sqrt{10}}\ \frac{8}{\sqrt{65}} + \frac{1}{\sqrt{10}}\ \frac{1}{\sqrt{65}} = \frac{5}{\sqrt{26}}$$

Example : 27

A man observes two objects in a straight line in the west. On walking a distance c to the north, the object subtend as angle α in front of him and on walking a further distance of c to the north, they subtend an angle β . Prove that the distance between the objects is : $\frac{3c}{2\cot\beta - \cot\alpha}$

Solution

Let x = distance between objects A and B.

y = distance of B from initial position of man.

The man starts from O and observes angle α and β at P and Q respectively as shown.

$$\alpha = \tan^{-1}\frac{x+y}{c} - \tan^{-1}\frac{y}{c}$$

$$\beta = \tan^{-1}\frac{x+y}{2c}\ \tan^{-1}\frac{y}{2c}$$

$$\Rightarrow \quad \tan\alpha = \frac{\frac{x+y}{c} - \frac{y}{c}}{1+\frac{xy+y^2}{c^2}} = \frac{ac}{c^2+xy+y^2}$$

$$\Rightarrow \quad \tan\beta = \frac{\frac{x+y}{2c} - \frac{y}{2c}}{1+\frac{xy+y^2}{4c^2}} = \frac{2xc}{4c^2+xy+y^2}$$

By eliminating $(xy + y^2)$, we can find x.
Equate values of $(xy + y^2)$ from the two equations.

$$\Rightarrow \quad \frac{xc}{\tan\alpha} - c^2 = \frac{2xc}{\tan\beta} - 4c^2$$

$$\Rightarrow \quad xc(\cot\alpha - 2\cot\beta) = -3c^2$$

$$\Rightarrow \quad x = \frac{3c}{2\cot\beta - \cot\alpha}$$

Example : 28

A right circular cylinder tower of height h and radius r stands on a horizontal lane. Let A be a point in the horizontal plane and PQR be the semi-circular edge of the two of the tower such that Q is the point in it nearest A. The angles of elevation of the point P and Q from A are 45º and 60º respectively. Show that :

$$\frac{h}{r} = \frac{\sqrt{3}(1+\sqrt{5})}{2}$$

Solution

Let P′, Q′, R′ be the projection of P, Q, R in the base of the lower. Hence PP′, QQ′, RR′ are vertical lines.
From $\Delta AQQ'$ $\quad AQ' = h\cot 60^\circ$
From $\Delta APP'$ $\quad AP' = h\cot 45^\circ$
If O is the centre of the circular base of the lower, triangle $\Delta AOP'$ is right angled

$$(h\cot 60^\circ + t)^2 + r^2 = (h\cot 45^\circ)^2$$

$$\Rightarrow \quad \frac{h^2}{3} + r^2 + \frac{2hr}{\sqrt{3}} + r^2 = h^2$$

$$\Rightarrow \quad 2h^2 - 2\sqrt{3}\,hr - 6r^2 = 0$$

$$\Rightarrow \quad \frac{h}{r} = \frac{2\sqrt{3} + \sqrt{12+48}}{4} \qquad \text{(taking only position values of } \frac{h}{r}\text{)}$$

$$\Rightarrow \quad \frac{h}{r} = \frac{\sqrt{3}(1+\sqrt{3})}{2}$$

Example : 29

From a point on the horizontal plane, the elevation of the top of the hill is α. After walking a metres towards the summit up a slope inclined at an angle β to the horizontal, the angle of elevation is γ. Find the height of the hill.

Solution

Let PQ = h = height of the hill.
P is the top of the hill (summit)
At A, on the ground level, elevation of P is α
at B(AB = a) elevation of P is γ. AB is inclined at β to the horizontal
Let NQ = y
from ΔPAQ : $\quad AQ = h\cot\alpha$
from ΔPBN : $\quad BN = (h - y)\cot\gamma$
from ΔBAM : $\quad AM = a\cos\beta$
$BM = y = a\sin\beta$
But $\quad AQ = AM = BM$

$$\Rightarrow \quad h\cot\alpha = a\cos\beta + (h + y)\cot\gamma$$

$$\Rightarrow \quad h\cot\alpha = a\cos\beta + (h - a\sin\alpha)\cot\gamma$$

$$\Rightarrow \quad h = \frac{a\cos\beta - a\sin\beta\cot\gamma}{\cot\alpha - \cot\gamma}$$

$$\Rightarrow \quad h = \frac{2[\cos\beta\sin\alpha\sin\gamma - \sin\beta\cos\gamma\sin\alpha]}{\sin\gamma\cos\alpha - \cos\gamma\sin\alpha}$$

$$\Rightarrow \quad h = \frac{a\sin\alpha\sin(\gamma-\beta)}{\sin(\gamma-\alpha)}$$

Example : 30

Due south of a tower which is leaving towards north, there are two stations at distances x, y respectively from its foot. If α and β are the angles of elevation of the top of the tower at these station respectively, show that the inclination of the tower to the horizontal is given by : $\cot^{-1}\left(\frac{y\cot\alpha - x\cot\beta}{y-x}\right)$

Solution

Let PQ be the tower and θ be its inclination with the horizontal. At A, elevation of the top is α and at B, the elevation is β

Let PM is perpendicular to the ground and PM = h

from ΔPQM : MQ = h cot θ

from ΔPAM : AM = h cot α

from ΔPBM : BM = h cot β

$\Rightarrow$ AM – QM = x $\Rightarrow$ h cot α – h cot = x ....(i)

$\Rightarrow$ BM – QM = y $\Rightarrow$ h cot β – h cot θ = y ....(ii)

dividing (i) by (ii), we get

$$\Rightarrow \quad \frac{\cot\alpha - \cot\theta}{\cot\beta - \cot\theta} = \frac{x}{y}$$

$$\Rightarrow \quad \cot\theta = \frac{y\cot\alpha - x\cot\beta}{y-x}$$

$$\Rightarrow \quad \theta = \cot^{-1}\left[\frac{y\cot\alpha - x\cot\beta}{y-x}\right]$$

Example : 31

The width of a road is b feet. On one side of the road, there is a pole h feet high. On the other side, there is a building which subtends an angle θ at the top of the pole. Show that the height of the building is

$$\frac{(b^2+h^2)\sin\theta}{b\cos\theta + h\sin\theta}.$$

Solution

Let PQ = y be the height of the building

Let AB = h be the height of the pole.

Let $\angle$QAB = α = $\angle$AQP

from ΔAOB :

AQ = $\sqrt{b^2+h^2}$ and $\sin\alpha = \frac{b}{\sqrt{b^2+h^2}}$, $\cos\alpha = \frac{h}{\sqrt{b^2+h^2}}$

in ΔAPQ

$\angle$APQ = π = (θ + α)

using the sine rule in this triangle

$$\frac{y}{\sin\theta} = \frac{AQ}{\sin\angle APQ}$$

$$\Rightarrow \quad \frac{y}{\sin\theta} = \frac{\sqrt{b^2+h^2}}{\sin(\pi - \overline{\theta+\alpha})}$$

$$\Rightarrow \quad y = \frac{\sqrt{b^2+h^2}\sin\theta}{\sin\theta\cos\alpha + \cos\theta\sin\alpha}$$

$\Rightarrow \quad y = \dfrac{\sqrt{b^2+h^2}\sin\theta}{\dfrac{h}{\sqrt{b^2+h^2}}\sin\theta + \dfrac{b}{\sqrt{b^2+h^2}}\cos\theta}$

$\Rightarrow \quad y = \dfrac{(b^2+h^2)\sin\theta}{h\sin\theta + b\cos\theta}$

Example : 32

The angle of elevation of a tower at a point A due south of it is 30º. AT a point b due east of A, the elevation is 18º. If AB = a, show that the height of the tower is : $\dfrac{a}{\sqrt{2+2\sqrt{5}}}$

Solution

Let PQ = h be the height of the tower.

At A, due to south of it, the elevation is $\angle PAQ = 30^o$

At B, due east of A, the elevation is $\angle PBQ = 18^o$

from ΔPAQ : AQ = h cot 30º

from ΔPBO : BQ = h cot 18º

Now consider the right angled triangle ΔAQB in the horizontal plane :

$AQ^2 + AB^2 = BQ^2$

$h^2 \cot^2 30^o + a^2 = h^2 \cot^2 18^o$

$\Rightarrow \quad h = \dfrac{a}{\sqrt{\cot^2 18^o - \cot^2 30^o}}$

we have $\cot^2 30^o = 3$ and $\cot 18^o = 5 + 2\sqrt{5}$ (try to calculate it yourself)

$\Rightarrow \quad h = \dfrac{a}{\sqrt{2+2\sqrt{5}}}$

Example : 33

A circular plate of radius a touches a vertical wall. The plate is fixed horizontally at a height b above the ground. A lighted candle of length c stands vertically at the centre of the plate. Prove that the breadth of the shadow thrown on the wall where it meets the horizontal ground is : $\dfrac{2a}{c}\sqrt{b^2+2bc}$

Solution

Let r be the radius of the circle formed by the shadow of the plate on the ground

Length of candle = PQ = c

$\dfrac{r}{a} + \dfrac{c+b}{c}$

$\Rightarrow \quad r = \dfrac{a}{c}(c+b)$

let AB be the shadow cut by the vertical wall.

$\Rightarrow \quad AB = \sqrt{r^2 - a^2} = 2\sqrt{\dfrac{a^2}{c^2}(c+b)^2 - a^2}$

$\Rightarrow \quad AB = \dfrac{2a}{c}\sqrt{(c+b)^2 - c^2} = \dfrac{2a}{c}\sqrt{b^2+2bc}$

Example : 34

A man standing south of a lamp-post observes his shadow on the horizontal plane to be 24 feet long. On walking eastward a distance of 300 feet, he finds that his shadow is now 30 feet. If his height is 6ft, find the height of the lamp above the horizontal plane.

Solution

Let PQ be the lamp-post and AB be the man in his initial position. He moves from AB to A′B′.

$\Rightarrow$ AA′ = 300ft and AX = 24ft

initial length of the shadow = AX = 24ft.

final length of the shadow = A′Y′ = 30ft

ΔQXP ~ ΔBXA

$$\Rightarrow \quad \frac{PQ}{AB} = \frac{PX}{AX} \quad \Rightarrow \quad \frac{h}{6} = \frac{24+PA}{24}$$

$\Rightarrow$ PA = 4 h – 24

ΔQYP ~ D′YA′

$$\Rightarrow \quad \frac{PQ}{A'B'} = \frac{PY}{A'Y} \quad \Rightarrow \quad \frac{h}{6} = \frac{30+PA'}{30}$$

$\Rightarrow$ PA′ = 5h – 30

Apply Pythagoras Theorem in ΔPAA′ :

$\Rightarrow$ $PA^2 + AA'^2 = PA^{\alpha}$

$\Rightarrow$ $(4h - 24) + 300^2 = (5h - 30)^2$

$\Rightarrow$ $9(h - 6)^2 = 300^2$

$\Rightarrow$ h = 106 ft.

Example : 35

An object is observed from three points A, B, C in the same horizontal line passing through the base of object. The angle of elevation at B is twice and at C is thrice that at A. If AB = a, BC = b, prove that the height of the object is : $\frac{a}{2b}\sqrt{(a+b)(3b-a)}$.

Solution

Let PQ be tower of height h.

Let θ , 2θ and 3θ be the angles of elevations of Q at A, B and C respectively

ΔQAB is isosceles $\Rightarrow$ QB = a

from ΔPQC; $QC = \frac{h}{\sin 3\theta}$

Applying sine rule in ΔQBC :

$$\Rightarrow \quad \frac{a}{\sin(\pi - 3\theta)} = \frac{b}{\sin\theta} = \frac{h/\sin 3\theta}{\sin 2\theta}$$

$$\Rightarrow \quad \frac{a}{\sin 3\theta} = \frac{b}{\sin\theta} = \frac{h}{\sin 3\theta \sin 2\theta}$$

$$\Rightarrow \quad \frac{a}{\sin 3\theta} = \frac{b}{\sin\theta}$$

$\Rightarrow$ $a = b(3 - 4\sin^2\theta)$

$\Rightarrow$ $\sin^2\theta = 3b - a$

$$\Rightarrow \quad \cos^2\theta = 1 - \left(\frac{3b-a}{4b}\right) = \frac{b+a}{4b}$$

$$\Rightarrow \quad \frac{a}{\sin 3\theta} = \frac{h}{\sin 3\theta \sin 2\theta}$$

$\Rightarrow$ $h = a \sin 2\theta$

$\Rightarrow$ $h = 2a \sin\theta \cos\theta$

$\Rightarrow \quad h = 2a \sqrt{\dfrac{3b-a}{4b}} \sqrt{\dfrac{b+a}{4b}}$

$\Rightarrow \quad h = \dfrac{a}{2b} \sqrt{(3b-a)(b+a)}$

Example : 36

A flagstaff on the top of a tower is observed to subtend the same angle α at two points on a horizontal plane, which lie on a line passing through the centre of the base of tower and whose distance from one another is 2a and angle β at a point half-way between them. Prove that the height of flagstaff is :

$a \sin \alpha \sqrt{\dfrac{2\sin\beta}{\cos\alpha \sin(\beta-\alpha)}}$

Solution

Let PQ be the tower and QR be the flagstaff. Let QR = 2h and PN = y

QR subtends α at A and B (where N is the mid-point of QR)

$\Rightarrow$ Q, R, A, B are concyclic. Let O be the centre of circle passing through these points.

$\Rightarrow \quad \angle POQ = 2\alpha$ (angle subtended at centre is double)

from ΔNOR : ON = h cot α

OR = radius = h cosec α

Let M be the mid-point of AB where QR subtends β

Let PM = x = ON

$\Rightarrow \quad x = h \cot \alpha$ ....(i)

from ΔOBM : $OM^2 = OB^2 - a^2 = h^2 \operatorname{cosec}^2\alpha - a^2$

$\Rightarrow \quad y^2 = h^2 \operatorname{cosec}^2\alpha - a^2$ ....(ii)

Now $\beta = \tan^{-1} \dfrac{PR}{PM} - \tan^{-1} \dfrac{PQ}{PM} = \tan^{-1} \dfrac{y+h}{x} = \tan^{-1} \dfrac{y-h}{x}$

$\Rightarrow \quad \tan \beta = \dfrac{\dfrac{y+h}{x} - \dfrac{y-h}{x}}{1 + \dfrac{y^2-h^2}{x^2}} \quad \Rightarrow \quad \tan \beta = \dfrac{2hx}{x^2+y^2-h^2}$

From (i), (ii) and (iii), we will eliminate x and y to get h.

$\Rightarrow \quad \tan \beta \,(h^2 \cot^2\alpha + h^2 \operatorname{cosec}^2\alpha - a^2 - h^2) = 2h\,(h \cot \alpha)$

$\Rightarrow \quad \tan \beta \,(2h^2 \cot^2\alpha - a^2) = 2h^2 \cot \alpha$

$\Rightarrow \quad h^2 = \dfrac{a^2 \tan\beta}{2\tan\beta\cot^2\alpha - 2\cot\alpha} = \dfrac{a^2 \sin\beta\sin^2\alpha}{2\sin\beta\cos^2\alpha - 2\cos\beta\cos\alpha\sin\alpha}$

$\Rightarrow \quad h^2 = \dfrac{a^2 \sin\beta\sin^2\alpha}{2\cos\alpha\sin(\beta-\alpha)} \quad \Rightarrow \quad h = a \sin \alpha \sqrt{\dfrac{\sin\beta}{2\cos\alpha\sin(\beta-\alpha)}}$

$\Rightarrow \quad$ height of flagstaff = 2h = $a \sin \alpha \sqrt{\dfrac{2\sin\beta}{\cos\alpha\sin(\beta-\alpha)}}$