Example : 1

(a) Show that the vectors $3\hat{i} + 6\hat{j} - 12\hat{k}$ and $2\hat{i} + 4\hat{j} - 8\hat{k}$ are collinear

(b) Show that the points A(1, 3, 2), B(–2, 0, 1) and C(4, 6, 3) are collinear

Solution

Let $\vec{a} = 3\quad + 6\hat{j} - 12\hat{k}$

$\vec{b} = 2\quad + 4\hat{j} - 8\hat{k}$

$\Rightarrow$ $\vec{a} = 3(\quad + 2\hat{j} - 4\hat{k}) = 3/2 \qquad (2\hat{i} + 4\hat{j} - 8\hat{k})$

$\Rightarrow$ $a = 3/2\ \vec{b}$

$\Rightarrow$ and are collinear

(b) $= (-2 - 1) \quad + (0 - 3)\ \hat{j} + (1 - 2)\ \hat{k}$

$\Rightarrow$ $\overrightarrow{AB} = -3 \quad - 3\hat{j} - \hat{k}$

$\overrightarrow{AC} = (4 - 1)\ \hat{i} + (6 - 3)\ \hat{j} + (3 - 2)\ \hat{k}$

$\Rightarrow$ $\overrightarrow{AC} = -3\hat{i} + 3\hat{j} + \hat{k}$

$\Rightarrow$ $\overrightarrow{AC} = -\overrightarrow{AB}$

$\Rightarrow$ vectors and are collinear

$\Rightarrow$ points A, B, C are collinear

Example : 2

(a) Show that the vectors $\hat{i} - \hat{j} - 2\hat{k}$, $2\hat{i} + 3\hat{j} + \hat{k}$, and $7\hat{i} + 3\hat{j} - 4\hat{k}$ are coplanar

(b) Show that the points P (a + 2b + c), Q (a – b – c), R (3a + b + 2c) and S (5a + 3b + 5c) are coplanar given that $\vec{a}$, , are non-coplanar. $\overrightarrow{AB}$

Solution

(a) Let $a = \quad - \hat{j} - 2\hat{k}$

$b = 2\hat{i} - 3\hat{j} + \hat{k}$

$c = 7\hat{i} + 3\hat{j} - 4\hat{k}$

Let $\vec{a} = \lambda \quad + \mu$ where λ and μ are scalars

If there exists unique values of λ and μ satisfying the above equation than we can say that can be expressed as a linear combination of and and hence , , are coplanar.

$= \lambda \qquad + \mu$

$\Rightarrow$ $\quad - \hat{j} - 2\hat{k} = \lambda(2\hat{i} + 3\hat{j} - \hat{k}) + \mu\ (7\hat{i} + 3\hat{j} - 4\hat{k})$

$\Rightarrow$ $\hat{i} - \hat{j} - 2\hat{k} = (2\lambda + 7\mu)\ \hat{i} + (3\lambda + 3\mu)\ \hat{j} + (\lambda - 4\mu)\ \hat{k}$

As $\hat{i}, \hat{j}, \hat{k}$ are non-coplanar, by equating coefficients of $\hat{i}, \hat{j}, \hat{k}$ we get :

$1 = 2\lambda + 7\mu$ ....(i)

$-1 = 3\lambda + 3\mu$ ....(ii)

$-2 = \lambda - 4\mu$ ....(iii)

Solving (i) and (ii), we get $\lambda = -2/3$ and $\mu = 1/3$

Substituting λ, μ in (iii), we have $-2 = -2/3 - 3\ (1/3)$

$\Rightarrow$ $-2 = -2$

$\Rightarrow$ λ, μ satisfy (iii)

$\Rightarrow$ $\lambda = 2/3$, $\mu = 1/3$ is a unique solution of $\vec{a} = \lambda \quad + \mu$

⇒ = – 2/3 + 1/3

As can be uniquely expressed as a linear combination of and , vectors , , are coplanar.

(b) To show that P, Q, R, S are coplanar, we will show that **PQ, PR, PS** are coplanar

PQ = – 3 – 2

PR = – 2 – +

PS = 4 + + 4

Let = λ**PR** + μ**PS**

⇒ – 3b – 2c = λ (2 – +) + μ (4 + + 4)

⇒ –3 – 2 = (2λ + 4μ) + (–λ + u) + (λ + 4μ)

As the vectors a, b, c are non-coplanar, we can equate their coefficients

⇒ 0 = 2λ + 4μ

⇒ – 3 = –λ + μ

⇒ – 2 = λ + 4μ

λ = 2, μ = – 1 is the unique solution for the above system of equations

⇒ **PQ** = 2**PR** – **PS**

PQ, PR, PS are coplanar because **PQ** is a linear combination of **PR** and **PS**

⇒ the points P, Q, R, S are also coplanar

Example : 3

Prove that the segment joining mid-points of the diagonals of a trapezium is parallel to the parallel sides of the trapezium and is equal to half the difference of their lengths.

Solution

Let ABCD be the given trapezium and M, N be the mid points of the diagonals AC and BD

Let , , , be the position vectors of A, B, C, D respectively

using section formula, mid points of AC and BD (BC) – AD $\frac{(\vec{b}+\vec{d})}{2}$ 2 2

= , $\vec{n}$ = ,

⇒ **NM** = $\vec{m} - \vec{n}$ =

⇒ **NM** = $\left(\frac{\vec{c}-\vec{b}}{2}\right)$ –

⇒ **NM** = 1/2 (**BC** – **AD**)

Let **BC** = k (**AD**)

⇒ **NM** = 1/2 (k – 1) **AD**

NM || AD and NM = 1/2 (k – 1) AD

⇒ NM = = $\frac{BC - AD}{2}$

⇒ NM is parallel to AD (and BC) and is half the difference of BC and AD.

Example : 4

Show that the diagonals of a parallelogram bisect each other

Solution

Let $\vec{a}$ + + + be the position vectors of a vertices of a parallelogram ABCD

AB = BC and AB || DC (because ABCD is a parallelogram)

⇒ **AB** = **DC**

⇒ – = –

⇒ (+)/2 = (+)/2

⇒ pv of mid point of BD = pv of mid point of AC

⇒ mid points of BD and AC coincide. Hence AC and BD bisect each other

Example : 5

Show that the medians of the triangle are concurrent and the point of concurrence divides each median in the ratio 2 : 1.

Solution

Let + + be the position vectors of the vertices of a triangle ABC

Let D, E, F be the mid-points of sides as shown

⇒ d = (+)/2 ⇒ 2 = +

⇒ e = (+) /2 ⇒ 2 = +)

⇒ f = (+) /2 ⇒ 2 = +

Now try to make the RHS of each equation equal

⇒ 2 + = + +

⇒ 2 + = + +

⇒ 2 + = + +

⇒ 2 + = 2 + = 2 + = + +

Note that the sum of scalar coefficients of vectors is equal to 3 in each expression. We divide each term by 3

⇒ $= \frac{2\vec{c}+\vec{b}}{2} = \frac{2\vec{f}+\vec{c}}{3} = \frac{2\vec{f}+\vec{c}}{3} = \frac{\vec{a}+\vec{b}+\vec{c}}{3}$

⇒ $\frac{2\vec{d}+\vec{a}}{2+1}$ = = =

⇒ the point G [(+ +)/3] divides AB, BE and CF each internally in ratio 2 : 1. Hence G is the common point of intersection of all medians.

⇒ medians are concurrent and centroid G divides each median in 2 : 1

Centroid G ≡

Example : 6

Show that the angle bisectors of a triangle are concurrent and hence find the position vector of the incentre.

Solution

Let + + be the position vectors of the vertices and x, y, z be the lengths of the sides opposite to these vertices respectively. Let AD, BE, CF be the angle bisectors.

Let us first find the position vector of D where AD bisects angle A.

$= \frac{z}{y}$ (bisector theorem)

using section formula

$\vec{d}$ =

Similarly, the position vectors of E and F are :

, $\vec{f} = \frac{-y\vec{b}+x\vec{a}}{y+x}$

$\Rightarrow$ (z + y) $\vec{d}$ = z + y

$\Rightarrow$ (x + z) = x + z

$\Rightarrow$ (y + x) = y + x

Make the RHS of each equation equal to x + y + z

$\Rightarrow$ (z + y) + x + z + y

$\Rightarrow$ (x + z) + y = y + x + z

$\Rightarrow$ (y + x) + z = z + y + x

$\Rightarrow$ (z + y) + x = (x + z) + y = (y + x) + z = x + y + z

$\Rightarrow$ = = =

$\Rightarrow$ the point I

i.e. =

divides each internal bisector (AD, BE, CF) in the ratio + y : x, x + z : y + x : z internally
Hence I is the common point of intersection of all bisectors i.e. the incentre

Example : 7

D, E divides sides BC and CA of a triangle ABC in the ratio 2 : 3 respectively. Find the position vector of the point of intersection of AD and BE and the ratio in which this point divides AD and BE.

Solution

Using section formula

[illegible]

$$\vec{e} = \frac{2\vec{a} + 3\vec{c}}{2 + 3}$$

$\Rightarrow$ 5$\vec{d}$ = 2 + 3

(Try to make RHS equal)

$\Rightarrow$ 5 = 2 + 3

$\Rightarrow$ 15 = 6 + 9 $\Rightarrow$ 15 + 4 = 4 + 6 + 9

$\Rightarrow$ 10 = 4 + 6 $\Rightarrow$ 10 + 9 = 9 + 4 + 6

15 + 4 = 10 + 9 + 6

Divide by the sum of coefficients of vectors (≡19)

$\Rightarrow$ = =

$\Rightarrow$ the point R i.e. $R \equiv \left(\frac{4\vec{a} + 9\vec{b} + 6\vec{c}}{19}\right)$

divides AD and BE internally in 15 : 4 and 10 : 9 respectively

Example : 8

Show that the segments joining vertices to the centroid of opposite faces are concurrent. Hence find the position vector of the point of concurrent

Solution

Let a, b, c, d be the position vectors of vertices of the tetrahedron and P, Q, R, S be the centroid of faces opposite to A, B, C, D respectively

⇒

⇒ $3\vec{q} = \vec{c} + \vec{d} + \vec{a}$

⇒ $3\vec{r} = \vec{d} + \vec{a} + \vec{b}$

⇒ $\vec{p} = \dfrac{\vec{b}+\vec{c}+\vec{d}}{3}$, $\vec{q} = \dfrac{\vec{c}+\vec{d}+\vec{a}}{3}$, $\vec{r} = \dfrac{\vec{d}+\vec{a}+\vec{b}}{3}$, $\vec{s} = \dfrac{\vec{a}+\vec{b}+\vec{c}}{3}$

⇒ $\dfrac{3\vec{p}+\vec{a}}{4} = \dfrac{3\vec{q}+\vec{b}}{4} = \dfrac{3\vec{r}+\vec{c}}{4} = \dfrac{3\vec{s}+\vec{d}}{4} = \dfrac{\vec{a}+\vec{b}+\vec{c}+\vec{d}}{4}$

⇒ $G \equiv \left(\dfrac{\vec{a}+\vec{b}+\vec{c}+\vec{d}}{4}\right)$

G is the common point of intersection of AP, BQ, CR, DS divides each in the ratio 3 : 1 internally

Example : 9

Derive the following results in a triangle using vectors.

(i) Since Rule

(ii) Cosine Rule

(iii) Projection Formula

Solution

Consider a triangle ABC, where

= **BC** , = **CA** , = **AB**

⇒ ....(i)

Note that the angle between $\vec{a}$ and is (π – C)

(i) Sine Rule :

Using (i)

$\vec{a}$ ×

× = × ....(ii)

Using (ii)

+ (+ +) = ×

⇒

⇒ $\vec{b} \times \vec{c} = \vec{a} \times \vec{b}$ ....(iii)

Combining (ii) and (iii), we get,

$\vec{a} \times \vec{b} = \vec{b} \times \vec{c} = \vec{c} \times \vec{a}$

⇒ $|\vec{a} \times \vec{b}| = |\vec{b} \times \vec{c}| = |\vec{c} \times \vec{a}|$

⇒ ab sin (π – C) = bc sin (π – A) = ca sin (π – B)

Divide by abc, we get :

⇒ $\dfrac{\sin C}{c} = \dfrac{\sin A}{a} = \dfrac{\sin B}{b}$

which is the sine rule

(ii) Using (i)

$\vec{a} + \vec{b} = -\vec{c}$

$(\vec{a} - \vec{b})^2 = \vec{c}^2$

$\Rightarrow$ $a^2 + b^2 + 2ab \cos(\pi - C) = c^2$

$\Rightarrow$ $a^2 + b^2 - 2ab \cos C = c^2$ which is the cosine rule

(iii) Using (i)

$\vec{a} + \vec{b} = -\vec{c}$

$\Rightarrow$ $\vec{a} \cdot \vec{c} + \vec{b} \cdot \vec{c} = -\vec{c} \cdot \vec{c}$

$\Rightarrow$ $ac \cos(\pi - B) + bc \cos(\pi - A) = -c^2$

$\Rightarrow$ $a \cos B + b \cos A = c$

which is the projection formula

Example : 10

Show that

(i) $\sin(\alpha + \beta) = \sin\alpha \cos\beta + \cos\alpha \sin\beta$

(ii) $\sin(\alpha - \beta) = \sin\alpha \cos\beta - \cos\alpha \sin\beta$

Solution

Consider a circle of radius r centred at origin

Consider two points P and Q, one in first (i) quadrant and other in fourth (iv) quadrant such that $\angle POX = \beta$ and $\angle QOX = \alpha$

$\Rightarrow$ $P \equiv (r \cos\beta, -r \sin\beta)$

$\Rightarrow$ $Q \equiv (r \cos\alpha, r \sin\alpha)$

OP = OQ = $(r \cos\beta\, i - r \sin\beta\, j) \times (r \cos\alpha\, i + r \sin\alpha\, j)$

Comparing magnitude,

$\Rightarrow$ **|OP| |OQ|** $\sin(\alpha + \beta) = r^2 (\sin\alpha \cos\beta + \cos\alpha \sin\beta)$

$\Rightarrow$ $\sin(\alpha + \beta) = \sin\alpha \cos\beta + \cos\alpha \sin\beta$ $\vec{a}^2 + \vec{b}^2 + 2\vec{a} \cdot \vec{b} = \vec{c}^2$

Following the same figure but considering OP . OQ

we can derive the formula for $\cos(\alpha + \beta)$

(ii) Now let us take both P and Q in Q, as shown

$\Rightarrow$ $P \equiv (r \cos\beta, r \sin\beta)$

$\Rightarrow$ $Q \equiv (r \cos\alpha, r \sin\alpha)$

OP × **OQ** = $(r \cos\beta\, i + r \sin\beta\, j) \times (r \cos\alpha\, i + r \sin\alpha\, j)$

Comparing magnitudes

$\Rightarrow$ **|OP| |OQ|** $\sin(\alpha - \beta) = r^2 (\sin\alpha \cos\beta - \cos\alpha \sin\beta)$

$\Rightarrow$ $\sin(\alpha - \beta) = \sin\alpha \cos\beta - \cos\alpha \sin\beta$ (OP = OQ = r)

Following the same figure but considering OP . OQ, we can derive the formula for $\cos(\alpha - \beta)$

Example : 11

Show that the angle in a semi-circle is a right angle

Solution

Let O be the centre and r be the radius of the semi-circle

Let **OP** = **QO** = $\vec{a}$ and **OR** =

$\Rightarrow$ **QR** = + and **RP** = –

Now **QR** . **RP** = (+) . (–) = $a^2 - b^2 = a^2 - b^2 = 0$

because a = b = radius of the semi-circle

Example : 12

The vertices of a triangle are A(2, 3, 0), B(–3, 2, 1) and C(4, –1, 0). Find the area of the triangle ABC and unit vector normal to the plane of this triangle.

Solution

Area of ΔABC = 1/2 |AB × AC|

AB = (–3 – 2) i + (2 – 3) j + (1 – 0) k

⇒ AB = –5 i – j + k

and **AC** = 2i – 4j + 0k

AB × **AC** = = 4i + 2j + 22k

⇒ area of ΔABC = $\frac{1}{2}\sqrt{16+4+484} = \sqrt{126}$ sq. units

$$= \frac{\overrightarrow{AB}\times\overrightarrow{AC}}{|\overrightarrow{AB}\times\overrightarrow{AC}|} = \frac{4\hat{i}+2\hat{j}+22\hat{k}}{2\sqrt{126}} = \frac{2\hat{i}+\hat{j}+11\hat{k}}{\sqrt{126}}$$

Example : 13

A line makes angles α, β, γ and δ with the diagonals of a cube. Prove that :

$\cos^2\alpha + \cos^2\beta + \cos2\gamma + \cos^2\delta = 4/3$

Solution

Let the origin O be one of the vertices of the cube and OA, OB, OC be the edges through O along the axes so that :

OA = a i,

OB = a j,

OC = a k

where a is the length of the edge of the cube. Let P, Q, R, S be the other vertices of the cube opposite to O, A, B, C respectively.

Hence the diagonals of the cube are OP, AQ, BR and CS.

$$\frac{\hat{n}\cdot\overrightarrow{OP}}{|\overrightarrow{OP}||\hat{n}|} \qquad \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -5 & -1 & 1 \\ 2 & -4 & 0 \end{vmatrix}$$

OP = a i + a j + a k

AQ = – a i + a j + a k

BR = a i – a j + a k

$\overline{CS}$ = a i + a j – a k

If $\vec{n}$ = x i + y j + z k is the unit vector along the line which makes the angles α, β, γ and δ with diagonals,

$$\cos\alpha = \quad = \frac{ac+ay+az}{a\sqrt{3}} = \frac{x+y+z}{\sqrt{3}}$$

$$\cos\beta = \frac{-x+y+z}{\sqrt{3}}; \qquad \cos\gamma = \frac{x-y+z}{\sqrt{3}} \qquad \cos\delta = \frac{x+y-z}{\sqrt{3}}$$

⇒ $\cos^2\alpha + \cos^2\beta + \cos^2\gamma + \cos^2\delta$

$$\frac{1}{3}\left[(x+y+z)^2 + (-x+y+z)^2 + (x-y+z)^2 + (x+y-z)^2\right]$$

$$\frac{1}{3}\,4\,(x^2+y^2+z^2) = 4/3 \quad [\because x^2+y^2+z^2 = 1]$$

Example : 14

Let A_r (r = 1, 2, 3, 4) be the areas of the faces of a tetrahedron. Let n, be the outward normals to the respectively faces with magnitude equal to corresponding areas. Prove that $\vec{n}_1 + \vec{n}_2 + \vec{n}_3 + \vec{n}_4 = 0$

Solution

Let the origin O be one vertex of the tetrahedron and A, B, C be the other vertices whose position vectors are :

OA = , **OB** = , **OC** =

$\vec{n}_2 = Ar(\Delta\ OBC) = 1/2\ (\vec{b} \times \vec{c})$

$\vec{n}_3 = Ar(\Delta\ OCA) = 1/2\ (\vec{c} \times \vec{a})$

$\vec{n}_4 = Ar(\Delta\ ABC) = 1/2$ (**AC** × **AB**)

⇒ [taken outwards from tetrahedron]

= 1/2 $(\vec{c} \times \vec{b} + \vec{a} \times \vec{c} + \vec{b} \times \vec{a})$

⇒ $\vec{n}_1 + \vec{n}_2 + \vec{n}_3 + \vec{n}_4 = \vec{0}$

Example : 15

Show that : $\vec{a} = (\vec{a}\,.\,\hat{i})\,\hat{i} + (\vec{a}\,.\,\hat{j})\,\hat{j} + (\vec{a}\,.\,\hat{k})\,\hat{k}$

Solution

Let $\vec{a} = a_x,\ \hat{i} + a_y\hat{j} + a_z\hat{k}$ ....(i)

⇒

⇒ and [illegible]

On substituting the values of a_x, a_y and a_z in (i), we get :

⇒

Example : 16

Show that : $\hat{i} \times (\vec{a} \times \hat{i}) + \hat{j} \times (\vec{a} \times \hat{j}) + \hat{k} \times (\vec{a} \times \hat{k}) = 2\vec{a}$

Solution

LHS = $[(\hat{i}\,.\,\hat{i})\,\vec{a} - (\hat{i}\,.\,\vec{a})\,\hat{i}] + [(\hat{j}\,.\,\hat{j})\,\vec{a} - (\hat{j}\,.\,\vec{a})\,\hat{j}] + [(\hat{k}\,.\,\hat{k})\,\vec{a} - (\hat{k}\,.\,\vec{a})\,\hat{k}]$

= $\vec{a} + \vec{a} + \vec{a} - [(\hat{i}\,.\,\vec{a})\,\hat{i} + (\hat{j}\,.\,\vec{a})\,\hat{j} + (\hat{k}\,.\,\vec{a})\,\hat{k}]$

= $3\vec{a} - [a_x\hat{i} + a_y\hat{j} + a_z\hat{k}]$ = R.H.S.

Note : It is useful to remember that x-component of etc.

Example : 17

(i) If the four points a, b, c, d are coplanar, then show that : |a b c| + |b c d| + |c a d| + |a b d|

(ii) If a, b, c, d are any four vectors then prove that :

$(\vec{a} \times \vec{b})\,.\,(\vec{c} \times \vec{d}) + (\vec{b} \times \vec{c})\,.\,(\vec{a} \times \vec{d}) + (\vec{c} \times \vec{a})\,.\,(\vec{b} \times \vec{d}) = 0$

Solution

(i) If the four points A , B , C and D are coplanar, then vectors **AB**, **AC**, **AD** are copla-nar

⇒ **AB** . **AC** × **AD** = 0

⇒

$\Rightarrow$ $= 0$

$\Rightarrow$ $\vec{b}\cdot\vec{c}\times\vec{d}+\vec{b}\cdot\vec{a}\times\vec{c}+\vec{b}\cdot\vec{d}\times\vec{a}-\vec{a}\cdot\vec{c}\times\vec{d}=0$

$\Rightarrow$

$\Rightarrow$ $[\vec{b}\ \vec{c}\ \vec{d}]+[\vec{a}\ \vec{b}\ \vec{d}]+[\vec{c}\ \vec{a}\ \vec{d}]=[\vec{a}\ \vec{b}\ \vec{c}]$

Note that interchanging two adjacent letters reverses the sign of scalar triple product.

$\Rightarrow$ [b d a] = – [b a d] = + [a b d]

(ii) Let us first simplify (a × b) . (c × d) taking p = a × b

(a × b) . (c × d) = p . c × d = (p × c) . d = {(a × b × c} . d

Interchanging dot and cross has no effect on scalar triple product of p, c and d.

$= [(\vec{a}\cdot\vec{c})\,\vec{b}-(\vec{b}\cdot\vec{c})\,\vec{a}]\cdot\vec{d}$

$= (\vec{a}\cdot\vec{c})\,(\vec{b}\cdot\vec{d})-(\vec{b}\cdot\vec{c})\,(\vec{a}\cdot\vec{d})$

Similarly, we can show that :

$(\vec{b}\times\vec{c})\cdot(\vec{a}\times\vec{d})=(\vec{b}\cdot\vec{a})(\vec{c}\cdot\vec{d})-(\vec{c}\cdot\vec{a})\,(\vec{b}\cdot\vec{d})$

$(\vec{c}\times\vec{a})\cdot(\vec{b}\times\vec{d})=(\vec{c}\cdot\vec{b})(\vec{a}\cdot\vec{d})-(\vec{a}\cdot\vec{b})\,(\vec{c}\cdot\vec{d})$

Adding the three equations, we get

$(\vec{a}\times\vec{b})\cdot(\vec{c}\times\vec{d})+(\vec{b}\times\vec{c})\cdot(\vec{a}\times\vec{d})+(\vec{c}\times\vec{a})\cdot(\vec{b}\times\vec{d})=0$

Example : 18

Show that : $= [\vec{a}\ \vec{b}\ \vec{c}]^2 =$

$\begin{vmatrix} [illegible] & & \\ \vec{b}.\vec{a} & \vec{b}.\vec{b} & \vec{b}.\vec{c} \\ \vec{c}.\vec{a} & \vec{c}.\vec{b} & \vec{c}.\vec{c} \end{vmatrix}$

Solution

Let us first prove $= [\vec{a}\ \vec{b}\ \vec{c}]^2$

L.H.S. = $.\ (\vec{b}\times\vec{c})\ \times\ (\vec{c}\times\vec{a})$

Let p = $(\vec{b}\times\vec{c})$

$\Rightarrow$ L.H.S. $= \vec{a}\times\vec{b}\ \{\vec{p}\times(\vec{c}\times\vec{a})\}$

$= \vec{a}\times\vec{b}\ \{(\vec{p}\cdot\vec{a})\,\vec{c}-(\vec{p}\cdot\vec{c})\,\vec{a}\}$

$= \vec{a}\times\vec{b}\ \{[\vec{b}\ \vec{c}\ \vec{a}]\,\vec{c}-0\}$

$=$

$= [\vec{b}\ \vec{c}\ \vec{a})\ [\vec{a}\ \vec{b}\ \vec{c}]$

$=$

Now let us show that :

$=$

$$\text{L.H.S.} = \quad = \begin{vmatrix} a_x & a_y & a_z \\ b_x & b_y & b_z \\ c_x & c_y & c_z \end{vmatrix} \begin{vmatrix} a_x & a_y & a_z \\ b_x & b_y & b_z \\ c_x & c_y & c_z \end{vmatrix} = \begin{vmatrix} \sum a_x^2 & \sum a_x b_x & \sum a_x c_x \\ \sum b_x a_x & \sum b_x^2 & \sum b_x c_x \\ \sum c_x a_x & \sum c_x b_x & \sum c_x^2 \end{vmatrix}$$

$$= \begin{vmatrix} \vec{a}.\vec{a} & \vec{a}.\vec{b} & \vec{a}.\vec{c} \\ \vec{b}.\vec{a} & \vec{b}.\vec{b} & \vec{b}.\vec{c} \\ \vec{c}.\vec{a} & \vec{c}.\vec{b} & \vec{c}.\vec{c} \end{vmatrix} = \text{R.H.S.}$$

Example : 19

If a.b ≠ 0, find the vector r which satisfies the equations : and $\vec{r} \,.\, \vec{a} = 0$

Solution

It is given that,

$\Rightarrow \quad \vec{r} \times \vec{b} = \vec{c} \times \vec{b}$

Taking cross product with a on both sides.

$\vec{a} \times (\vec{r} \times \vec{b}) = \vec{a} \times (\vec{c} \times \vec{b})$

$(\vec{a} \,.\, \vec{b})\vec{r} - (\vec{a} \,.\, \vec{r})\, \vec{b} = (\vec{a} \,.\, \vec{b})\, \vec{c} - (\vec{a}.\vec{c})\, \vec{b}$

$\Rightarrow \quad (\vec{a} \,.\, \vec{b})\vec{r} = (\vec{a} \,.\, \vec{b})\, \vec{c} - (\vec{a} \,.\, \vec{c})\, \vec{b}$ (using $\vec{r} \,.\, \vec{a} = 0$)

$\Rightarrow$ $(\because\ \vec{a} \,.\, \vec{b} \neq 0)$

Alternative method

$\Rightarrow \quad (\vec{r} - \vec{c})$ and $\vec{b}$ are collinear

$\begin{matrix} b_x & b_y & b_z \\ c_x & c_y & c_z \end{matrix}$

$\Rightarrow$ where k is a scalar

$\Rightarrow \quad \vec{r} = \vec{c} + k\, \vec{b}$ ....(i)

it is given that $\vec{r} \,.\, \vec{a} = 0$

$\Rightarrow$

$\Rightarrow$

$\Rightarrow \quad k = -$

using (i), we get $\vec{b}$

Example : 20

Express in terms of $\vec{a}, \vec{b}, \vec{c}$. It is given that are non-coplanar

Solution

Let

We take dot product with b × c on both sides

$\Rightarrow \quad (\vec{a} \times \vec{b}) \,.\, (\vec{b} \times \vec{c}) = \ell \vec{a} \,.\, \vec{b} \times \vec{c} + 0 + 0$

$\Rightarrow \quad \ell =$

Similarly by taking dot product with c × a and a × b, we get

$m = \dfrac{(\vec a \times \vec b) \,.\, (\vec c \times \vec a)}{[\vec c\ \vec a\ \vec b]}$ and $n = \dfrac{(\vec a \times \vec b) \,.\, (\vec a \times \vec b)}{[\vec a\ \vec b\ \vec c]}$

$$\Rightarrow \quad \vec a \times \vec b = \frac{1}{[\vec a\ \vec b\ \vec c]}\ [(\vec a \times \vec b) \,.\, (\vec b \times \vec c)\, \vec a + (\vec a \times \vec b) \,.\, (\vec c \times \vec a)\, \vec b + (\vec a \times \vec b) \,.\, (\vec a \times \vec b)\, \vec c]$$

Following the same pattern, we can express b × c and c × a in terms of a, b, c

Example : 21

Find a vector of magnitude 5 units coplanar with vectors 3i – j – k and i + j – 2k and perpendicular to the vector 2i + 2j + k.

Solution

Let $\vec a = 3i - j - k$

and

A vector coplanar with and and perpendicular to can be taken as

where ℓ is a scalar

$$\vec a \times \vec b = \begin{vmatrix} \hat i & \hat j & \hat k \\ 3 & -1 & -1 \\ 1 & 1 & -2 \end{vmatrix} = 3\hat i + 5\hat j + 4\hat k$$

$\sqrt{\ldots 25 + 15}$ $5\sqrt{2}$ $[\vec b\ \vec c\ \vec a]$

$$\vec c \times (\vec a \times \vec b) = \begin{vmatrix} \hat i & \hat j & \hat k \\ 2 & 2 & 1 \\ 3 & 5 & 4 \end{vmatrix} = 3\hat i - 5\hat j + 4\hat k$$

$= \vec r = \ell(3i - 5j + 4k)$

$= 5$

$\ell = \pm \qquad = \pm \dfrac{1}{\sqrt{2}}$

the required vector is $\vec r = \pm \dfrac{1}{\sqrt{2}}(3\hat i - 5\hat j + 4\hat k)$

Alternate Method

As the required vector $\vec r$ is coplanar with and , we take

$\Rightarrow \quad \vec r = (3\lambda + \mu)\, i + (-\lambda + \mu)\, j - (-\lambda - 2\mu)\, k$

As is perpendicular to ,

$\Rightarrow \quad 2(3\lambda + \mu) + 2(-\lambda + \mu) + (-\lambda - 2\mu) = 0$

$\Rightarrow \quad 3\lambda + 2\mu = 0$

$\Rightarrow \quad \lambda = (-2/3)\, \mu$

We also have

$\Rightarrow$ = 5

$\Rightarrow \quad \mu^2 \left(1 + \frac{25}{9} + \frac{16}{9}\right) = 25$

$\Rightarrow \quad \mu = \pm \frac{3}{\sqrt{2}} \qquad \Rightarrow \qquad \lambda = \pm \frac{2}{\sqrt{2}}$

$\Rightarrow \quad \vec{r} = \pm \frac{1}{\sqrt{2}}(3\hat{i} - 5\hat{j} + 4\hat{k})$

Example : 22

Show that the lines $\vec{r}$ = 3i – j + k + λ (i + j + k) and = 2i + 2j – 2k + μ (i – j + 2k) are intersecting and hence find their point of intersection.

Solution

Let be the position vector of their point of intersection

$\Rightarrow$ = 3i – j + k + λ (i + j + k) = 2i + 2j – 2k + μ (i + j + 2k)

$\Rightarrow$ (3 + λ) i + (λ – 1) j + (λ + 1) k = (μ + 2) i + (2 – μ) j + (2μ – 2) k

$\Rightarrow$ 3 + λ = μ + 2 ....(i)

$\Rightarrow$ λ – 1 = 2 – μ ....(ii)

$\Rightarrow$ λ + 1 = 2μ – 2 ....(iii)

The lines are intersecting if these equations are consistent

from (i) and (ii), we get

λ = 1, μ = 2

Substituting these values in (iii), we get

1 + 1 = 2 (2) – 2

$\Rightarrow$ 2 = 2

$\Rightarrow$ λ = 1, μ = 2 satisfied (iii) also $\sqrt{(3\lambda+\mu)^2 + (\mu-\lambda)^2 + (-\lambda-2\mu)^2}$ (b – a) . (c – b)

Hence lines are intersecting and the point of intersection is :

= 3i – j + k + λ (i + j + k)

= 3i – j + k + 1 (i + j + k)

= 4i + 2k

$\Rightarrow$ the coordinates of this point are (4, 0, 2)

Example : 23

The vertices of a triangle ABC are A(1, 0, 2), B(–2, 1, 3) and C(2, –1, 1). Find the equation of the line BC, the foot of the perpendicular from A to BC and the length of the perpendicular.

Solution

A vector parallel to BC is

BC = = 4i – 2j – 2k

$\Rightarrow$ the equation of BC is : $\vec{r} = \vec{b} + t\,(\vec{c} - \vec{b})$

$\Rightarrow$ $\vec{r}$ = – 2i + j + 3k + t (4i – 2j – 2k)

Let position vector of D be = – i + j + 2k + t (4i – 2j – 2k)

because D lies on line BC

Now AD ⊥ BC $\Rightarrow$ **AD . BC** = 0

$\Rightarrow$ = 0

$\Rightarrow$ [–3i + j + k + t(4i – 2j – 2k)] . (4i – 2j – 2k) = 0

$\Rightarrow$ (4t – 3) 4 + (1 – 2t) (–2) + (1 – 2t) (–2) = 0

$\Rightarrow$ 24t – 16 = 0

$\Rightarrow$ t = 2/3

$\Rightarrow \quad \vec{d} = -2i + j + 3k + (2/3)(4i - 2j - 2k)$

$=$

$\Rightarrow \quad D \equiv \left(\frac{2}{3}, -\frac{1}{3}, \frac{5}{3}\right)$

$\vec{AD} = \quad = \left(\frac{2}{3} - 1\right) i - \frac{1}{3} j + \left(\frac{5}{3} - 2\right) k = -\frac{1}{3} i - \frac{1}{3} j - \frac{1}{3} k$

$AD = |AD| = \frac{\sqrt{3}}{3} = \frac{1}{\sqrt{3}}$ units

Example : 24

Find the equation of the plane passing through the points A(2, 1, 3), B (–1, 2, 4) and C (0, 2, 1). Hence find the coordinate of the point of intersection of the plane ABC and the line r = i – j + k + λ (2i + k).

Solution

Let $\vec{N}$ be a vector normal to the plane of ΔABC

$\Rightarrow \quad = \mathbf{AB} \times \mathbf{AC}$

$= (-3i + j + k) \times (-2i + j - 2k)$

$= \quad = -3i - 8j - k$

$\Rightarrow$ the equation of the plane is $(\vec{r} - \vec{a}) \cdot \vec{N} = 0$ where $= 2i + j + 3k$

$\frac{2}{3}\hat{i} - \frac{1}{3}\hat{j} + \frac{5}{3}\hat{k}$

$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -3 & 1 & 1 \\ -2 & 1 & -2 \end{vmatrix}$

$\Rightarrow \quad \vec{r} \cdot (-3i - 8j - k) = -6 - 8 - 3$

$\Rightarrow \quad \cdot (-3i - 8j - k) = -17$

The given line is $= i - j + k + \lambda (2i + k)$

To find the point of intersection, we solve these equations simultaneously

$\Rightarrow \quad [i - j + k + \lambda (2i + k)] - (-3i - 8j - k) = -17$

$\Rightarrow \quad (2\lambda + 1)(-3) + 8 + (l + 1)(-1) = -17$

$\Rightarrow \quad \lambda = 3$

$\Rightarrow$ the point of intersection is $= i - j + k + 3(2i + k)$

$= 7i - j + 4k$

$\Rightarrow$ coordinates are (7, –1, 4)

Example : 25

From the point A (1, 2, 0), perpendicular is drawn to the plane r . (3i – j + k) = 2 meeting it at the point P. Find the coordinates of point P and the distance AP.

Solution

Let us first find the equation of line AP. As AP is normal to the plane, the vector $= 3i - j + k$ is parallel to AP.

$\Rightarrow$ Equation of AP is $= i + 2j + t(3i - j + k)$

Now we solve equations of AP and plane to get point P.

$\Rightarrow \quad [i + 2j + t(3i - j + k)] \cdot (3i - j + k) = 2$

$\Rightarrow \quad (3t + 1)\,3 + (2 - t)(-1) + t = 2$

$\Rightarrow \quad t = 1/11$

$\Rightarrow$ point P is $= i + 2j + 1/11\,(3ii - j + k)$

=

$\Rightarrow \qquad P \equiv \left(\frac{14}{11}, \frac{21}{11}, \frac{1}{11}\right)$

$$P = \sqrt{\left(\frac{14}{11} - 1\right)^2 + \left(\frac{21}{11} - 2\right)^2 + \left(\frac{1}{11} - 0\right)^2} = \frac{1}{\sqrt{11}}$$

Example : 26

Find the shortest distance between the non-intersecting and non-parallel (SKEW) lines :

$\vec{r} = \vec{a}_1 + \lambda \vec{b}_1$ and $\vec{r} = \vec{a}_2 + \mu \vec{b}_2$

Solution

Let PQ be the shortest length segment between the lines.
PQ is hence perpendicular to the lines at P and Q
PQ = projection of A_1A_2 on the normal vector of both lines.
PQ = projection of $A_1 A_2$ on $\vec{b}_1 \times \vec{b}_2$

$$PQ = \frac{(\vec{a}_2 - \vec{a}_1) \cdot (\vec{b}_1 \times \vec{b}_2)}{|\vec{b}_1 \times \vec{b}_2|}$$

Example : 27

If $\vec{A} = x_1\vec{a} + y_1\vec{b} + z_1\vec{c}$; $\vec{B} = x_2\vec{a} + y_2\vec{b} + z_2\vec{c}$; $\vec{C} = x_3\vec{a} + y_3\vec{b} + z_3\vec{c}$, prove that $\vec{A} \cdot \vec{B} \times \vec{C} = \begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix}$

$(\vec{a} \cdot \vec{b} \times \vec{c})$ $\qquad \vec{A} \cdot \vec{B} \times \vec{C}$ $\frac{14}{11}\hat{i} + \frac{21}{11}\hat{j} + \frac{1}{11}\hat{k}$

Solution

Let $\vec{a} = a_1i + a_2j + a_3k$; $= b_1i + b_2j + b_3k$ and $= c_1i + c_2j + c_3k$

$\Rightarrow \qquad = (x_1 a_1 + y_1b_1 + z_1c_1)\, i + (x_1 a_2 + y_1b_2 + z_1c_2)\, j + (x_1 a_3 + y_1 b_3 + z_1c_3)\, k$

$= (x_2 a_1 + y_2b_1 + z_2c_1)\, i + (x_2 a_2 + y_2b_2 + z_2c_2)\, j + (x_2 a_3 + y_2 b_3 + z_2c_3)\, k$ and

$= (x_3 a_1 + y_3b_1 + z_3c_1)\, i + (x_3 a_2 + y_3b_2 + z_3c_2)\, j + (x_3 a_3 + y_3 b_3 + z_3c_3)\, k$

Therefore,

$$= \begin{vmatrix} x_1a_1 + y_1b_1 + z_1c_1 & x_1a_2 + y_1b_2 + z_1c_2 & x_1a_3 + y_1b_3 + z_1c_3 \\ x_1a_1 + y_2b_1 + z_2c_1 & x_2a_2 + y_2b_2 + z_2c_2 & x_2a_3 + y_2b_3 + z_2c_3 \\ x_3a_1 + y_3b_1 + z_3c_1 & x_3a_2 + y_3b_2 + z_3c_2 & x_3a_3 + y_3b_3 + z_3c_3 \end{vmatrix}$$

$$= \begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_3 \\ x_3 & y_3 & z_3 \end{vmatrix} \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = \begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix} (\vec{a} \cdot \vec{b} \times \vec{c})$$

Example : 28

ABC is a triangle. E and F are the midpoints at AC and AB respectively. CP is drawn parallel to AB to meet BE. produced in P. Show that ΔFEP = ΔFCE = 1/4 ΔABC

Solution

Let C be the origin; p.v.'s of A and B be $\vec{a}$ and respectively.

p.v. of E = and p.v. of F = $\vec{f} = \dfrac{\vec{a}+\vec{b}}{2}$

Since CP || AB,

$\therefore$ p.v. of P can be $\mu\ (\vec{b} - \vec{a})$ ....(i)

Also equation of line BE is $\vec{r} = \vec{b} + \vec{t}(\vec{b} - \vec{e})$

$\Rightarrow$ $\vec{r} = \vec{b} + \vec{t}\left(\vec{b} - \dfrac{\vec{a}}{2}\right)$

$\Rightarrow$ $\vec{r}$ = (–t/2) + (1 + t)

As P lies on BE, prove that of P should satisfy above equation :

μ = (–t/2) $\vec{a}$ + (1 + t) [using (i)]

On comparing the coefficients of vectors and , we get

$\Rightarrow$ t = 1 = t/2 $\Rightarrow$ t = – 2

and μ = – 1

Therefore, prove that of P is

Vector area (ΔFEP) $= \dfrac{1}{2}\ \overrightarrow{FE} \times \overline{EP} = \dfrac{1}{2}(\vec{e} - \vec{f}) \times (\vec{p} - \vec{c})$

$= \dfrac{1}{2}\left(\dfrac{\vec{a}}{2} - \dfrac{\vec{a}}{2} - \dfrac{\vec{b}}{2}\right) \times$ $= \dfrac{1}{2}\left(\dfrac{\vec{b}}{2}\right) \times$

= $\vec{a} \times \vec{b}$ ....(ii)

Again vector Area (ΔFCE) $= \dfrac{1}{2}\ \overrightarrow{FC} \times \overrightarrow{CE}$ = [illegible] $(\vec{f}) \times (\vec{e})$ [illegible] $= 2t\,\vec{b} \times \vec{b} = \vec{0}$

$= \left(-\dfrac{\vec{a}}{2} - \dfrac{\vec{b}}{2}\right) \times$ = $\vec{a} \times \vec{b}$ ......(iii)

From (ii) and (iii),

Vector Area (ΔFEP) = Vector Area (ΔFCE) = 1/8 $\vec{a} \times \vec{b}$ = 1/4 Vector Area (ΔABC)

Example : 29

If one diagonal of a quadrilateral bisects the other, then it also bisects the quadrilateral.

Solution

Let OABC be the given quadrilateral such that its diagonal **OB** bisects the diagonal **AC**.

Let **OA** = $\vec{a}$, **OB** = and **OC** = .

The mid-point of $= \dfrac{\vec{a}+\vec{c}}{2}$.

As diagonals bisect each other, p.v. of OB = p.v. of AC

$\Rightarrow$ p.v. of OB = $\dfrac{\vec{a}+\vec{c}}{2}$

Since OB is parallel to vector $\vec{b}$, we can take :

= t $\vec{b}$ $\Rightarrow$

Multiplying both sides with vector $\vec{b}$, we have

⇒ $\vec{a} \times \vec{b} = \vec{b} \times \vec{c}$

⇒ 1/2 $\vec{a}$ × = 1/2 | × |

⇒ Area of ΔOAB = Area of ΔOBC [∵ OABC is a plane quadrilateral]

Hence the diagonal OB bisects the quadrilateral.

Example : 30

Let OACB be a parallelogram with 0 at the origin and OC a diagonal. Let D be the midpoint of OA. Using vector methods prove that BD and CO intersect in the same ratio. Determine this ratio.

Solution

Method 1 :

Let P be the point of intersection of OC and BD.

Let be the p.v.'s of the points A, B, C, D, P, respectively

As OACB is a parallelogram and D is mid-point of OA, we can take :

and $\vec{c} = \vec{a} + \vec{b}$

Equation of line OC is : $\vec{r} = \lambda\,(\vec{a} + \vec{b})$ ....(i)

Equation of line BD is : $\vec{r} = \vec{b} + \mu(\vec{d} - \vec{b})$

$= \vec{b} + \mu\,(\vec{a}/2 - \vec{b})$

⇒ $\vec{r} = \mu\,\vec{a}/2 + (1 + \mu)\,\vec{b}$ ....(ii)

Solving (i) and (ii), we can get point of intersection P i.e.

$\lambda\vec{a} + \lambda\vec{b} = \mu\,\vec{a}/2 + (1 - \mu)\,\vec{b}$

⇒ λ = μ/2 and λ = 1 – μ (since $\vec{a}$ and are non collinear)

⇒ μ/2 = 1 – μ

⇒ μ = 2/3 and λ = 1/3 $\vec{p} = 1/3(\vec{a} + \vec{b})$

From (i),

⇒ **OP** = 1/3 **OC**

⇒ P divides CO in the ratio 2 : 1 ....(iii)

From (ii), $\vec{p} = \vec{b} + 2/3\,(\vec{d} - \vec{b})$

⇒ $\vec{p} - \vec{b} = 2/3\,(\vec{d} - \vec{b})$

⇒ **BP** = 2/3 **BD**

P divides BD in the same ratio 2 : 1 ....(iv)

From (iii) and (iv), we can conclude that BD and OC intersect in the same ratio.

Method 2

OD = 1/2 **OA** = 1/2 **BC**

⇒ **OP** + **PD** = 1/2 (**BP** + **PC**)

⇒ 2**OP** + 2 **PD** = **BP** + **PC**

⇒ –2 **PO** + 2**PD** = – **PB** + **PC**

⇒ **PB** + 2**PD** = **PC** + 2 **PO**

⇒ $\dfrac{\overrightarrow{PB} + 2\,\overrightarrow{PD}}{1+2} = \dfrac{\overrightarrow{PC} + 2\,\overrightarrow{PO}}{1+2}$

⇒ The common point P of BD and CO divides each in the ratio 2 : 1

Example : 31

A transversal cuts the sides OL, OM and diagonal ON of a parallelogram at A, B, C respectively.

Prove that $\dfrac{OL}{OA} + \dfrac{OM}{OB} = \dfrac{ON}{OC}$.

Solution

We have $\overrightarrow{ON} = \overrightarrow{OL} + \overrightarrow{LN} = \overrightarrow{OL} + \overrightarrow{OM}$ ....(i)

Let $\overrightarrow{OL} = x\,\overrightarrow{OA}$, $\overrightarrow{OM} = y\,\overrightarrow{OB}$ and $\overrightarrow{ON} = z\,\overrightarrow{OC}$ ....(ii)

So $\left|\overrightarrow{OL}\right| = x\left|\overrightarrow{OA}\right|, \left|\overrightarrow{OM}\right| = y\left|\overrightarrow{OB}\right|$ and $\left|\overrightarrow{ON}\right| = z\left|\overrightarrow{OC}\right|$

$\Rightarrow$ $x = \dfrac{OL}{OA}$, $y = \dfrac{OM}{OB}$ and $z = \dfrac{ON}{OC}$

From (i) and (ii), we have

$z\overrightarrow{OC} = x\,\overrightarrow{OA} + y\,\overrightarrow{OB}$

$\Rightarrow$ $x\,\overrightarrow{OA} + y\,\overrightarrow{OB} - z\,\overrightarrow{OC} = \vec{0}$

Point A, B, C are collinear, the sum of the coefficients of their p.v. must be zero

$\Rightarrow$ $x + y - z = 0$

i.e. $+ \dfrac{OM}{OB} = \dfrac{ON}{OC}$

Example : 32

D, E, F are the midpoint of the sides BC, CA, AB respectively of a triangle ABC and O is any point. Show that

(i) $\overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC} = \overrightarrow{OD} + \overrightarrow{OE} + \overrightarrow{OF}$

$(\vec{d} - \vec{a}) + (\vec{e} - \vec{b}) + (\vec{f} - \vec{c})$ $\overrightarrow{OE} + \overrightarrow{OF} + \overrightarrow{DO}$ $\dfrac{OL}{OA}$

(ii) $\overrightarrow{AD} + \overrightarrow{BE} + \overrightarrow{CF} = \vec{0}$

(iii) $\overrightarrow{AE} + \overrightarrow{OF} + \overrightarrow{DO} = \overrightarrow{OA}$

(iv) $\overrightarrow{AD} + \dfrac{2}{3}\overrightarrow{BE} + \dfrac{1}{3}\overrightarrow{CF} = \dfrac{1}{2}\overrightarrow{AC}$

Solution

(i) Considering the point O as origin and

$\overrightarrow{OD} + \overrightarrow{OE} + \overrightarrow{OF} = \vec{d} + \vec{c} + \vec{f} = \dfrac{\vec{b}+\vec{c}}{2} + \dfrac{\vec{c}+\vec{a}}{2} + \dfrac{\vec{a}+\vec{b}}{2}$

Hence $\overrightarrow{OD} + \overrightarrow{OE} + \overrightarrow{OF} = \overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC}$

(ii) $\overrightarrow{AD} + \overrightarrow{AE} + \overrightarrow{AF} =$

$= (\vec{d} + \vec{e} + \vec{f}) - (\vec{a} + \vec{b} + \vec{c}) = \vec{0}$ [using (i)]

(iii) Consider $= \vec{e} + \vec{f} - \vec{d}$

$= \dfrac{\vec{c}+\vec{a}}{2} + \dfrac{\vec{a}+\vec{b}}{2} + \dfrac{\vec{b}+\vec{c}}{2} = \vec{a} =$

(iv) $\overrightarrow{AD}\ \frac{2}{3}\overrightarrow{BE} + \frac{1}{3}\overrightarrow{CF} = \vec d - \vec a + 2/3\,(\vec e - \vec b) + 1/3\,(\vec f - \vec c)$

$$= \frac{\vec b + \vec c}{2} - \vec a + \frac{2}{3}\left(\frac{\vec c + \vec a}{2} - \vec b\right) + \frac{1}{3}\left(\frac{\vec a + \vec b}{2} - \vec c\right)$$

$$= \left(-1 + \frac{1}{3} + \frac{1}{6}\right)\vec a + \qquad \vec b + \qquad \vec c$$

=

Example : 33

Let $\vec a, \vec b$ and be three non-coplanar unit vectors, equally inclined to one another at an angle θ. If :

, find scalars p, q and r in terms of θ

Solution

$\vec a.\vec a = \vec b.\vec b = \vec c.\vec c = 1$ and

Consider

Taking dot product with $\vec a, \vec b$ and , we get

= (q + r) cos θ ....(i)

0 = q + (p + r) cos θ ....(ii)

= cos θ (p + q) + r ....(iii)

Add (i), (ii) and (iii) to get

2 = (1 + 2 cos θ) (p + q + r) ....(iv) [illegible]

Solving (i) and (iv), we get

p = ...(v)

Solving (ii) and (iv), we get

q = ...(vi)

Solving (iii) and (iv), we get

r = ...(vii)

Now find

= =

Apply $R_1 \rightarrow R_1 + R_2 + R_3$

$\Rightarrow \qquad [\vec{a}\,\vec{b}\,\vec{c}]^2 = (1 + 2\cos\theta)$

Apply $C_1 \to C_1 - C_2$ and $C_2 \to C_2 - C_3$

$\Rightarrow \qquad [\vec{a}\,\vec{b}\,\vec{c}]^2 = (1 + 2\cos\theta) \qquad = = (1 + 2\cos\theta)(1 - \cos\theta)^2$

$\Rightarrow \qquad [\vec{a}\,\vec{b}\,\vec{c}] = \qquad (1 - \cos\theta)$

Substituting the value of $[\vec{a}\,\vec{b}\,\vec{c}]$ in (v), (vi) and (vii) to get

p = , $q = \dfrac{-2\cos\theta}{\sqrt{1+2\cos\theta}}$, $r = \dfrac{1}{\sqrt{1+2\cos\theta}}$

Example : 34

Vectors $\vec{x}, \vec{y}$ and $\vec{z}$ each of magnitude $\sqrt{2}$, make angles of 60º with each other. If

and $\vec{x} \times \vec{y} = \vec{c}$, then find $\vec{x}, \vec{y}$ and z in terms of

Solution

and

It is given that : $\qquad \Rightarrow \qquad \vec{y} - \vec{z} = \vec{a}$ ....(i)

and $\vec{y} \times (\vec{z} \times \vec{x}) = \vec{b} \qquad \Rightarrow \qquad \vec{z} - \vec{x} = \vec{b}$ ....(ii)

On adding (i) and (ii), we get $\qquad \vec{y} - \vec{x} = \vec{a} + \vec{b}$ ....(iii)

$\vec{x}, \vec{y}$ ... $= \vec{a}$, $\vec{y} \times (\vec{z} \times \vec{x}) = \vec{b}$ ... $\cos\theta$ ... $\cos\theta - 1$ ... 1

consider $\vec{x} \times \vec{y} = \vec{c}$

Take cross-product with x to get

$\vec{x} \times (\vec{x} \times \vec{y}) = \vec{x} \times \vec{c} \qquad \Rightarrow \qquad \vec{x} - 2\,\vec{y} = \vec{x} \times \vec{c}$ ....(iv)

Take cross-product with y to get

$\vec{y} \times (\vec{x} \times \vec{y}) = \vec{y} \times \vec{c} \qquad \Rightarrow \qquad 2\vec{x} - \vec{y} = \vec{y} \times \vec{c}$ ....(v)

On subtracting (iv) from (v), we get ....(vi)

$\vec{x} + \vec{y} = (a + \vec{b}) \times \vec{c}$

On solving (iii) and (vii), we get ....(vii)

$\vec{y} = \dfrac{\vec{a} + \vec{b} + (\vec{a} + \vec{b}) \times \vec{c}}{2}$, $\vec{x} = -\left[\dfrac{\vec{a} + \vec{b} - (\vec{a} + \vec{b}) \times \vec{c}}{2}\right]$

Using (i), $\qquad \vec{z} = \dfrac{\vec{b} - \vec{a} + (\vec{a} + \vec{b}) \times \vec{c}}{2}$

Example : 35

The position vectors of the points P and Q are 5i + 7j – 2k and –3i + 3j + 6k respectively. The vector $\vec{A}$ = 3i – j + k passes through the point P and the vector = – 3i + 2j + 4k passes through the point Q. A third vector 2i + 7j – 5k intersects vectors and . Find the position vectors of the points of intersection.

Solution

Equation fo line AP ≡ = 5i + 7j – 2k + λ (3i – j + k)

Equation of line BQ ≡ r = – 3i + 3j + 6k + λ (–3i + 2j + k)

Since point D lies on AP, its position vector can be taken as : = r = 5i + 7j – 2k + λ_1 (3i – j + k)

A vector parallel to line CD is 2i + 7j – 5k

Equation of line CD ≡ r = 5i + 7j – 2k + λ_1 (3i – j + k) + λ (2i + 7j – 5k)

Solve equation of line BQ with equation of line CD to get point of intersection C.

Solve BQ and CD to get :

$$5i + 7j - 2k + \lambda_1 (3i - j + k) + \lambda_2 (2i + 7j - 5k) = -3i + 3j + 6k + \lambda_3 (-3i + 2j + k)$$

Equating the coefficients of i, j and k, we get

$$5 + 3\lambda_1 + 2\lambda_2 = -3 (1 + \lambda_3) \quad \text{............(i)}$$

$$7 - \lambda_1 + 7\lambda_2 = 3 + 2\lambda_3 \quad \text{............(ii)}$$

$$-2 + \lambda_1 - 5\lambda_2 = 6 + 4\lambda_3 \quad \text{............(iii)}$$

Solve equations (i), (ii) and (iii) to get :

$\lambda_2 = -1, \lambda_3 = -1$ and $\lambda_1 = -1$

⇒ D ≡ (2, 8, –3) and C ≡ (0, 1, 2)

$\vec{B}$ $\vec{A}$